Linear/ODE Lecture Notes

MATH 20580 • Spring 2024 • University of Notre Dame

notes by **Jack MapelLentz**
from lectures by **Dr. Henry Chimal-Dzul**

Contents

Note: you can click on the contents! And then there's also a button in the bottom right on each page to jump back here.

Lecture 1: systems of linear equations

Given a system of linear equations, one of the following is true:

- it has a unique solution,
- it has infinitely many solutions
- it has no solution at all.

An $m \times n$ matrix is an array with m rows and n columns where each entry is a real or complex number. Such a matrix is denoted by:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The notation for accessing elements is $(\text{row}, \text{column})$, so if we have $\begin{bmatrix} 1 & -1 & 2 \\ 3 & \pi & e \end{bmatrix}$, then the entry at $(2, 3)$ is e.

Every system of linear equations can be represented by a matrix called the augmented matrix, which of formed by appending the coefficient matrix of the system to the column vector of constantans. These are defined via an example in the next table.

Ways of representing a system of linear equations	
System of linear equations	$x + y - z = 3$ $2x + 5z = 1$ $x + 3y - 2z = 0$
Coefficient matrix	$\begin{bmatrix} x & y & z \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix}$
Augmented matrix	$[\text{coefficient matrix} \mid \text{constants}]$ $\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 3 \\ 2 & 0 & 5 & 1 \\ 1 & 3 & 2 & 0 \end{array}\right]$
Constant vector	$b = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$
Variable vector	$\bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
Matrix multuplication notation	$A \cdot \bar{x} = b$

Row-echelon form (REF) of a matrix

- The leading entry in each nonzero row is to the right of all the leading entries in the rows above it
- All all-zero rows are at the bottom

$$\begin{cases} x + y - z = 3 \\ -2y + 7z = -5 \\ 11z = -7 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -2 & 7 & -5 \\ 0 & 0 & 11 & -7 \end{array}\right]$$

Reduced row-echelon form (RREF) of a matrix

In addition to the criteria for row-echelon form:

- every pivot should be 1
- above & below each pivot must be zeros

Rank of a matrix

The rank of a matrix is the number of pivots in either row-echelon form or reduced row-echelon form (it's the same either way)

Examples of row-echelon/reduced row-echelon form and rank

Examples of above things			
matrix	row-echelon form?	reduced row-echelon form?	rank
$\begin{bmatrix} 2 & 4 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$	✓	×	2
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$	✓	×	3
$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	✓	×	2
$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	×	×	n/a
$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$	×	×	n/a
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	✓	✓	3
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	✓	✓	2

Gaussian elimination

Augmented matrix → elementary row operations → row echelon form, then do backward substitution

$$[A \mid b] \to \cdots \to \text{row-echelon form} + \text{backward sub}$$

Gauss-Jordan elimination
Augmented matris → elementary operations → reduced row-echelon form, then backward substitution

$$[A \mid b] \to \cdots \to \text{reduced row-echelon form} + \text{backward sub}$$

Elementary row operations notation

operation	*meaning*
$R_i \leftrightarrow R_j$	interchange row i and row j
$R_i \leftarrow kR_i$	Multiply row i by a nonzero constant
$R_i \leftarrow R_i + kR_j$	Add k times row j to row i (store the result in row i)

Checking for a unique solution
When rank = number of variables, the system has a unique solution.

Example of Gaussian elimination
Solve the following system using Gaussian elimination:

$$\begin{cases} x + y - z = 3 \\ 2x + 5z = 1 \\ x + 3y - 2z = 0 \end{cases}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 2 & 0 & 5 & 1 \\ -1 & 3 & -2 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 + -2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -2 & 7 & -5 \\ 0 & 4 & -3 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & -2 & 7 & -5 \\ 0 & 0 & 11 & -7 \end{array}\right]$$

The right most matrix is in row-echelon form!
The matrix's rank is 3, and the number of variables is also 3, so the system has a unique solution.
To do backward substitution, go from the bottom up:

$$11z = -7 \rightsquigarrow z = -\frac{7}{11}$$

(and just continue plugging in from there as you solve variables).

Lecture 3: Gaussian elimination and solutions of systems of linear equations

Unique solution

Example: Solve the system

$$\begin{cases} 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 - 2x_3 = -5 \end{cases}$$

using Gaussian elimination.

Solution:

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + -2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Now we're in row-echelon form! The corresponding equations now are

$$\begin{cases} x_1 - x_2 - 2x_3 = -5 \\ x_2 + x_3 = 3 \\ x_3 = 2 \end{cases}$$

Then (from plugging in from the bottom-up in those equations) the system has a unique solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Note: remember from last lecture that the system has a unique solution iff (if and only if) the number of variables equals the number of pivots in row-echelon form, which is the same thing as the rank of the matrix.

$$\text{unique solutions} \Leftrightarrow \#\text{ of variables} = \text{pivots in REF} = \text{rank}([A \mid b])$$

No solution

Example: Solve the system

$$\begin{cases} x_1 - x_2 + 2x_3 = 3 \\ x_1 + 2x_2 - x_3 = -3 \\ 2x_2 - 2x_3 = 1 \end{cases}$$

using Gaussian elimination.

Solution:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

But the last line in the matrix gives you $0 = 5$! That means the system is inconsistent, so it doesn't have a solution.

In general, if you have $[A \mid b] \to \left[\begin{smallmatrix} 0 & \dots \end{smallmatrix} \middle| \begin{smallmatrix} b \end{smallmatrix}\right]$ in row-echelon form (so $0 = b$ with $b \neq 0$), the system is inconsistent and doesn't have a solution.

Infinitely many solutions (parametric form)

Example: Solve the system

$$\begin{cases} w - x - y + 2z = 1 \\ 2w - 2x - y + 3z = 3 \\ -w + x - y = -3 \end{cases}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last row (all zeroes) isn't telling you it's inconsistent — it's just saying that there's redundant information. So then switching this back into equations, we have

$$\begin{cases} w - x - y + 2z = 1 \\ y - z = 1 \end{cases}$$

Note: Now only w and y have corresponding pivots. These are called leading variables. The remaining variables (x and z) are called free variables. These free variables will create parameters — and that means that the system will have infinitely many solutions. (Thus, the system must be written in parametric form.)

So! Putting it in parametric form:

1. Assign parameters to free variables: $x = t, z = s$ where $t, s \in \mathbb{R}$
2. Express the leading variables in terms of the free variables:

$$\begin{cases} w = x + y - 2z + 1 \\ y = z + 1 \end{cases} \rightsquigarrow \begin{cases} w = x + z + 1 - 2z + 1 = x - z + 2 \\ y = z + 1 \end{cases}$$

3. And then finally express the solution in parametric form:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t - s + 2 \\ t \\ s + 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

This is the parametric form of a system which has infinitely many solutions!

Summary: Theorem (Rank and Solutions of Systems of Linear Equations)

Given a system of linear equations, say $Ax = b$. We have the following: −The system is consistent and:

- it has a unique solution iff $\text{rank}([A|b]) = \#\ \text{variables}$
- if has infinitely many solutions iff $\text{rank}([A|b]) < \#\ \text{variables}$. In this case, the number of free variables is given by $\#\ \text{free variables} = \#\ \text{of variables} - \text{rank}([A|b])$
- The system is inconsistent in which case the REF or RREF of $[A|b]$ has a row of the form $[0\ \cdots\ 0|b]$ with $b \neq 0$

Linear combinations

A system may have solutions written in parametric form, like

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} s$$

If you'd like to generate a solution, you need to give t and s values and combine them — this is what we call a linear combination, and we say that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$$

Lecture 4: spanning sets

Consider the homogeneous system

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = 0 \\ 2x_1 + 2x_2 + x_4 = 0 \\ 3x_1 + x_2 + 2x_3 = 0 \end{cases}$$

Note: “homogenous” means all the equations are equal to zero

The third equation here is totally irrelevant, actually: it’s just the first equation plus the second equation, so it’s not independent. This system will have infinitely many solutions.

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} \underline{1} & -1 & 2 & -1 & 0 \\ 0 & \underline{4} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 and x_2 are leading variables, because they have pivots (underlined above) in their columns, while x_3 and x_4 are free variables (i.e. parameters).

If we set $x_3 = t, x_4 = s$, the collection of solutions is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} \frac{1}{4} \\ -\frac{3}{4} \\ 0 \\ 1 \end{bmatrix} s$$

This is a linear combination because t and s are in linear form.

The solution of the homogenous system is the set

$$\operatorname{span}\left(\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}\frac{1}{4}\\-\frac{3}{4}\\0\\1\end{bmatrix}\right) = \text{the collection of all linear combinations of } v_1, v_2 = \{v_1 t + v_2 s : t, s \in \mathbb{R}\}$$

Geometrically, we have 4 variables, so we're in four-dimensional space ($\mathbb{R}^4$). v_1 and v_2 can be thought of as vectors, and when you add them together, you get a plane. This plane (in $\mathbb{R}^4$ still) contains all the solutions to the system.

The important point is that any vector on this plane is a linear combination of v_1 and v_2. Moreover, any vector on the plane is a solution to the homogenous system.

Examples: Given a vector $v \in \mathbb{R}^n$, determine if $v \in \operatorname{span}(v_1, ..., v_k)$.

Example: Is $v = \begin{bmatrix}0\\-2\\1\\4\end{bmatrix} \in \operatorname{span}\left(\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}\frac{1}{4}\\-\frac{3}{4}\\0\\1\end{bmatrix}\right)$?

Solution: We would like to find $t, s \in \mathbb{R}$ such that $\begin{bmatrix}0\\-2\\1\\4\end{bmatrix} = \begin{bmatrix}-1\\1\\1\\0\end{bmatrix} t + \begin{bmatrix}\frac{1}{4}\\-\frac{3}{4}\\0\\1\end{bmatrix} s$.

To solve, let's combine the two vectors on the right, so:

$$\begin{bmatrix}0\\-2\\1\\4\end{bmatrix} = \begin{bmatrix}-t+\frac{1}{4}s\\t-\frac{3}{4}s\\t\\s\end{bmatrix}$$

Then it's obvious that $t = 1$ and $s = 4$. You can check the validity of this solution with the first two entries. So the answer to the original question is yes, because you can find a solution.

Example: Is $\begin{bmatrix}1\\2\\4\\0\end{bmatrix}$ in $\operatorname{span}\left(\begin{bmatrix}1\\3\\0\\1\end{bmatrix},\begin{bmatrix}3\\1\\2\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$?

We would like to find (if they exist) $x, y, z \in \mathbb{R}$ such that:

$$\begin{bmatrix}1\\2\\4\\0\end{bmatrix} = \begin{bmatrix}1\\3\\0\\1\end{bmatrix} x + \begin{bmatrix}3\\1\\2\\0\end{bmatrix} y + \begin{bmatrix}1\\1\\1\\1\end{bmatrix} z$$

So then we combine them into one vector again:

$$\begin{bmatrix}1\\2\\4\\0\end{bmatrix} = \begin{bmatrix}x+3y+z\\3x+y+z\\2y+z\\x+z\end{bmatrix} \Leftrightarrow \begin{cases}x+3y+z=1\\3x+y+z=2\\2y+z=4\\x+z=0\end{cases}$$

Now we have to solve a system of linear equations, so let's use matrices:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -13 \end{array}\right]$$

The bottom row here says that $0 = -13$, which is inconsistent and obviously can't be true, so the system has no solutions. ($\therefore$ the answer to the original question is no)

Also: it's a lot quicker if you just go straight from the vectors in the question to a matrix, which will be useful on an exam.

Note: if there are infinitely many solutions, the vector is in the span, and furthermore there are infinitely many linear combinations that satisfy that.

Section 3.1: matix operations

Here's a thing:

$$\text{two matrices are equal} \Leftrightarrow \begin{matrix} 1) \text{ they have the same size} \\ 2) \text{ corresponding entries are equal} \end{matrix}$$

For example, if

$$A = \begin{bmatrix} x & y-1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ a & b & 1 \end{bmatrix}$$

then it's possible to have A and B equal if $\begin{cases} x=1 \\ y-1=2 \\ a=2 \\ b=3 \end{cases}$. But it's not possible to have A or B equal to C because they have different sizes.

Addition and subtraction of matrices (entry-wise)

These operations are only defined iff the matrices have the same size.

For example, if $A = \begin{bmatrix} 1 & 4 & 0 \\ -2 & 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix}$ then $A + B = \begin{bmatrix} -2 & 5 & -1 \\ 1 & 6 & 7 \end{bmatrix}$.

Scalar multiplication

Example: if $A = \begin{bmatrix} 1 & 4 & 0 \\ -2 & 6 & 5 \end{bmatrix}$ and $k = 2$, then $kA = 2A = \begin{bmatrix} 2 & 8 & 0 \\ -4 & 12 & 10 \end{bmatrix}$.

Matrix multiplication

For matrices A and B, the product AB is defined iff the number of columns of A = the number of rows of B — so if A is size $m \times n$, B must be size $n \times p$. The size of the output will then be $m \times p$.

Example: if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightsquigarrow AB = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This is fine because A is 3×2 and B is 2×2. The output will be 3×2.

Lecture 5: matrix multiplication, inverses, and linear transformations

Matrix multiplication (cont.)

Matrix multiplication: $\underbrace{A}_{m\times n}\underbrace{B}_{n\times p} = A\cdot B$

Example: you go row (in the left matrix) times column (in the right matrix), like this

$$\begin{bmatrix}1 & 0 & 5\\ 0 & 2 & -1\end{bmatrix}\cdot\begin{bmatrix}-1 & 0\\ 1 & 1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}(1)(-1)+(0)(1)+5(2) & (1)(0)+(0)(1)+(5)(1)\\ (0)(-1)+(2)(1)+(-1)(2) & (0)(0)+(2)(1)+(-1)(1)\end{bmatrix} = \begin{bmatrix}9 & 5\\ 0 & 1\end{bmatrix}$$

Example: the same thing as above but reversing the order

$$\begin{bmatrix}-1 & 0\\ 1 & 1\\ 2 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 5\\ 0 & 2 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0 & -5\\ 1 & 2 & 4\\ 2 & 2 & 9\end{bmatrix}$$

You can be super fast at this by taking the row from the left matrix and lining it up as a column and distributing it along the right matrix!

Remarks: For $a, b \in \mathbb{R}$, then $ab = ba$ and $ab = 0 \Rightarrow a = 0$ or $b = 0$. However, for matrices, these two properties may be false: $AB \neq BA$ and $AB = 0 \nRightarrow A = 0$ or $B = 0$.

For example: $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Theorem: distributivity law

1. $A(B+C) = A\cdot B + A\cdot C$
2. $(B+C)A = B\cdot A + C\cdot A$

Special matrices

Special matrices	
Zero matrix	$\begin{bmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{bmatrix}$
Identity matrix	$I_n = \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix}$ (blank spaces are 0)
Diagonal matrix $\text{dig}(a_1, a_2, ..., a_n)$	$\begin{bmatrix}a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n\end{bmatrix}$
Scalar matrix	$kI_n = \begin{bmatrix}k & & \\ & \ddots & \\ & & k\end{bmatrix}$

Properties with special matrices

$A + 0 = A$

(0 is the zero matrix)

$AI_n = A$ and $I_nA = A$

Invertibility

We noted that $AB \neq BA$ in general, but there are cases where $AB = BA$. For example,

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This property is similar to $2 \cdot \frac{1}{2} = \frac{1}{2} \cdot 2 = 1$. 1 is the identity for the real numbers, and $2 \cdot \frac{1}{2}$ can be written as $2^1 \cdot 2^{-1}$.

So, the **definiton** of this property: a matrix A of size $m \times n$ is invertible if there exists B such that $AB = BA = I_n$. (Hence, $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is invertible.)

Recall that for any $a \in \mathbb{R}, a \neq 0$, there exists $b \in \mathbb{R}$ such that $ab = 1$. However, it is possible for a matrix A to satisfy the following:

- $A \neq 0$
- $AB \neq I_n$

$\therefore$ not all matrices are invertible.

For example, let's try to see if this one is invertible:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is impossible because the bottom-right entry in the identity matrix will never be able to be nonzero. So $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

But how do you know what matrices *are* invertible?

Theorem about invertibility for a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is invertible} \Leftrightarrow \det(A) = ad - bc \neq 0$$

and if A is invertible then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

($\det(A)$ *is the determinant of* A*)*

For example, let's determine whether these matrices are invertible:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \rightsquigarrow \det(A) = 1 \cdot 0 - 3 \cdot 2 \neq 0 \therefore A^{-1} = \frac{1}{-6} \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ -2 & -1 \end{bmatrix} \rightsquigarrow \det(A) = (1)(-1) + \left(\frac{1}{2}\right)(-2) = 0 \therefore \nexists A^{-1}$$

Theorem about invertibility for any $m \times n$ matrix (which we'll come back to later)

For A of size $m \times n$, A is invertible $\Leftrightarrow \det(A) \neq 0$

Gauss-Jordan method for inverses

Input: A of size $m \times n$

Steps:

- With $[A \mid I_n] \to \cdots \to [B \mid A^{\sim}]$ where B is the reduced row-echelon form of A,
  - if $B = I_n$ and $A^{\sim} = A^{-1}$, then A is invertible
  - if $B \neq I_n$, then A is not invertible

Example: find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$, if it exists.

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{2} R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

Now we have the matrix $[I_3 \mid A^{-1}]$, so A is invertible and $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$.

Solving $Ax = B$**:** Firstly, how do we solve the equation $3x = 6$?

1. $3x = 6$
2. Find 3^{-1}: $\frac{1}{3}$
3. $\frac{1}{3} 3x = \frac{1}{3} 6$
4. $1x = 2$
5. $x = 2$

Now for $Ax = B$ where A and B are matrices:

1. write things down again
2. Find A^{-1} (if it exists)
3. $A^{-1}(Ax) = A^{-1}b$ (be careful of order because matrices are nor communative)
4. $I_n x = A^{-1}b$
5. $x = A^{-1}b$

Example: solve $\begin{cases} x+z=2 \\ y=2z=-1 \\ -x+z=3 \end{cases}$

Using the coefficent matrix, we can represent this as $A \cdot \bar{x} = b$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Find the inverse of A:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

Then solve:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ -12 \\ 5 \end{bmatrix}$$

Hence, $x = -\frac{1}{2}, y = -6, z = \frac{5}{2}$.

Linear transformations

Definiton:

A linear transformation is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

1. $T(u + v) = T(u) + T(v)$
2. $T(ku) = kT(u)$

For example, a linear transformation might transform a plane into a line, but will never transform a plane into, say, a parabola (since that would require exponents etc).

Examples

(Examples may run onto the next page)

Example 1: $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} |x| \\ y+z \end{bmatrix}$. Is T a linear transformation?

Nope! $|x|$ is not a linear function. We can justify this because it doesn't satisfy either of the above conditions (though you only really have to show that one of them doesn't work):

For the fist condition, $T(u+v) \neq T(u) + T(v)$ because with $u = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$:

$$T(u+v) = T\left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T(u) = T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} |1| \\ -1 + (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$T(v) = T\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} |-1| \\ 1+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T(u) + T(v) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T(u+v)$$

For the second condition, $T(ku) \neq kT(u)$ with $u = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, k = -2$ because:

$$T(ku) = T\left(2\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$kT(u) = -2T\left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$T(ku) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} -2 \\ 0 \end{bmatrix} = kT(u)$$

Example 2: $S : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ y+2z \end{bmatrix}$. Is S a linear transformation?

Yes — both properties are linear transformations. To prove this...

...for the first condition, $T(u+v) = T(u) + T(v)$, let $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$. Then

$$S(u+v) = S\left(\begin{bmatrix} x+a \\ y+b \\ z+c \end{bmatrix}\right) = \begin{bmatrix} x+a-(y+b) \\ y+b+2(z+c) \end{bmatrix}$$

$$S(u) + S(v) = S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} x-y \\ y+2z \end{bmatrix} + \begin{bmatrix} a-b \\ b+2c \end{bmatrix} = \begin{bmatrix} x-y+a-b \\ y+2z+b+2c \end{bmatrix}$$

$$\therefore S(u+v) = S(u) + S(v)$$

...for the second condition, $S(ku) = kS(u)$, let $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3, k \in \mathbb{R}$. Then

$$S(ku) = S\left(k\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = S\left(\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}\right) = \begin{bmatrix} kx-ky \\ ky+2kz \end{bmatrix}$$

$$kS(u) = kS\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = k\begin{bmatrix} x-y \\ y+2z \end{bmatrix} = \begin{bmatrix} kx-ky \\ ky+2kz \end{bmatrix}$$

$$\therefore S(ku) = kS(u)$$

Theorem

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$T(\bar{x}) = A\bar{x}$$

A is an $m \times n$ matrix called "the matrix associated to the linear transformation".

For $S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+2z \end{bmatrix}$, (the below columns go x, y, z)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Lecture 7: subspaces

Section 3.5: subspaces

Definition of a subspace

A subspace of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:

1. $\bar{0} = (0, ..., 0) \in S$
2. $U, V \in S \Rightarrow U + V \in S$
3. $K \in \mathbb{R}, U \in S, KU \in S$

A subspace is in some way equivalent to the algebraic version of planes, lines, ..., through the origin in $\mathbb{R}^n$.

Examples

Subspaces of $\mathbb{R}^2$:

1. For a point at the origin, the subspace would be $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$
   - This set satisfies each of the three above conditions, so it's a subset of $\mathbb{R}^2$.
2. For a line $y = -x$ (you could use any line that goes through the origin)
   - This line can be understood as all scalar multiples of a single vector, and a simple way to write that is that $S = \left\{ k \begin{bmatrix} -1 \\ 1 \end{bmatrix} : k \in \mathbb{R} \right\}$, and that is equivalent to $\operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$
     - This also satisfies the conditions, so it's a valid subset of $\mathbb{R}^2$
3. For $\mathbb{R}^2$ itself: $S = \mathbb{R}^2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$
   - The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are enough to express every possibility in $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^3$:

1. $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$
2. $S = \text{line through the origin} = \operatorname{span}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right)$
3. $S = \text{a plane in } \mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \right)$
4. $S = \mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

Theorem

Any subspace S of $\mathbb{R}^n$ can be written as

$$S = \operatorname{span}(v_1, ..., v_k)$$

for some vectors $v_1, ..., v_k \in \mathbb{R}^n$.

Examples

Example: Is $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = 3y, z = -2y \right\}$ a subspace of $\mathbb{R}^3$?

Substituting the conditions $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, y \in \mathbb{R} \Rightarrow S = \operatorname{span}\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right)$$

$\therefore S$ is a subspace of $\mathbb{R}^3$.

Example: Is $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = 3y + 1, z = -2y\right\}$ a subspace of $\mathbb{R}^3$?

Substitute in the conditions in $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+1 \\ y \\ -2y \end{bmatrix} = \begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compare this to

$$\operatorname{span}\left(\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \left\{t\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : t, s \in \mathbb{R}\right\}$$

But there is no parameter in the second vector from when we substituted the conditions (i.e. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$). This suggests that S is not a subspace of $\mathbb{R}^3$, but now we need to go back and show that one of the conditions fails.

Let's try checking whether (1) from the definition above fails:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 3y+1 \\ y \\ -2y \end{bmatrix}$$

Looking at the second row, $y = 0$. However, it should also be that $0 = 3y + 1$. Plugging in $y = 0$, that gives us $0 = 3(0) + 1 \rightsquigarrow 0 = 1$. Obviously that's not true, so this fails. $\therefore$ S is not a subspace of $\mathbb{R}^3$.

We can also try checking (3) from the defintion of a subspace: let's choose 0 for k and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for u. Then ku should be in S. However, $0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin S$. (We already showed that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in S.)

Subspaces associated with matrices

Definition

Let A be an $m \times n$ matrix. There are n entries in each row, so m rows are in $\mathbb{R}^n$. Conversely, there are m entries in each column, so n columns are in $\mathbb{R}^m$

1. Row space of A: $\operatorname{row}(A) =$ spanned set by the rows of A
2. Column space of A: $\operatorname{col}(A) =$ spanned set by the columns of A
3. Null space: $\operatorname{null}(A) =$ solutions to $A\bar{x} = 0$ (which is a homogenous system).

Problems

Determine if a vector is in one of the given subspaces and find bases and dimensions of each subspace.

Examples

Given that $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$,

1. is $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\text{col}(A)$?

By the definiton of column space,

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \right)$$

Then we can say that

$$b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \text{col}(A) \Leftrightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

Turning this into a system of equations,

$$\begin{cases} 1 = x - y \\ 2 = y \\ 3 = 3x - 3y \end{cases} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

This system is consistent, $\therefore \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \text{col}(A)$.

But! To do this even faster, you can jumpt to the fact that $b \in \text{col}(A) \Leftrightarrow [A \mid b]$ is consistent.

2. is $b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ in $\text{col}(A)$?

Jumping to the matrix, we have

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 4 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

The last row makes this matrix inconsistent, $\therefore \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \notin \text{col}(A)$.

3. is $[3\ \ 1]$ in $\mathrm{row}(A)$?

By definition, $\mathrm{row}(A) = \mathrm{span}([1\ \ -1], [0\ \ 1], [3\ \ -3])$. Hence

$$[3\ \ 1] \in \mathrm{row}(A) \Leftrightarrow [3\ \ 1] = x[1\ \ -1] + y[0\ \ 1] + z[3\ \ -3] \Leftrightarrow \begin{cases} 3 = x + 0y + 3z \\ 1 = -x + y - 3z \end{cases}$$

Now putting that into a matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ -1 & 1 & -3 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

The system is consistent and has a solution, so $[3\ \ 1] \in \mathrm{row}(A)$. Furthermore, if we take $z = 0$ to get a particular solution, then $y = 4, x = 3$, and

$$[3\ \ 1] = 3[1\ \ -1] + 4[0\ \ 1] + 0[3\ \ -3]$$

Note: we can notice that the rows in A became columns in our augmented matrix above. So given $[3\ \ 1]$ and $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$, then we can form an augmented matrix by writing rows as columns: $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 4 \end{array}\right]$.

Definiton: given a matrix A, we define the transpose of A as

$$A^T = \begin{bmatrix} — & \text{col } 1 & — \\ — & \text{col } 2 & — \\ — & \text{col } 3 & — \end{bmatrix}$$

Notice that $\left(A^T\right)^T = A$. Hence, $b = [-] \in \mathbb{R}^n$ is in $\mathrm{row}(A) \Leftrightarrow [A^T \mid b^T]$ is consistent.

4. is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in $\mathrm{null}(A)$?

Recall that $\mathrm{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is in $\mathrm{null}\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}\right)$ if and only if:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ * \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ * \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ * \\ * \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Lecture 8: subspaces, bases, & dimension

Bases and dimension

Goal: identify for a subspace $S \subseteq \mathbb{R}^n$ such that vectors $v_1, ..., v_k$ so that

1. $\text{span}(v_1, ..., v_k) = S$
2. $v_1, ..., v_k$ are linearly independent
   - $v_1, ..., v_k$ are linearly independent if $a_1 v_1 + ... + a_k v_k = 0 \Rightarrow a_1 = 0, ..., a_k = 0$ is the only solution.
   - When there is more than one way to get a solution, the equations are linearly *dependent*.
   - In other terms, $v_1, ..., v_k$ are linearly independent $\Leftrightarrow Ax = 0$ has a unique solution.
     - A in here is formed like so:
       - $A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$
     - $Ax = 0$ has a unique solution if:
       - $[A \mid 0] \to \cdots \to [B \mid 0]$ $\text{rank}(B) = k = \#$ of given vectors where B is in row-echelon form

Example: are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ linearly independent?

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \text{rank}(B) = 2$$

Meanwhile, there were 3 vectors given. ∴ the given vectors are linearly dependent.
For a further example, this means any one vector depends on the others:

$$1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

So

$$\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Hence $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $S = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

Example: explain why $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbb{R}^3$.

Solution:

1. Are they linearly independent? Yes!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank} = 3 = \# \text{ columns} \Rightarrow Ax = 0 \text{ has a unique solution} \Rightarrow \begin{matrix} \text{vectors are} \\ \text{linearly independent} \end{matrix}$$

2. Is $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbb{R}^3$? Yes, because:

Given $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Definiton: the dimension of a subspace S of $\mathbb{R}^n$ is

$$\dim(S) = \#\text{ of vectors in a basis for } S$$

Examples:

1. $\dim(\mathbb{R}^2) = 2$ because $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis. This is called the standard basis.
2. $\dim(\mathbb{R}^3) = 3$ because a basis for $\mathbb{R}^3$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.
3. Given $S = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$, $\dim(S) = 2$ because $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ is a basis for S.

Lecture 9: bases for subspaces of matrices and coordinate systems in $\mathbb{R}^n$

Example: Given

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

find bases and dimensions for $\text{row}(A)$ $\text{col}(A)$, and $\text{null}(A)$.

Solution: in general, first compute the row-echelon form of the given matrix, then proceed to compute the bases and dimension.

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix} \xrightarrow{\text{to RREF}} B = \begin{bmatrix} \textcircled{1} & 0 & 1 & 0 & -1 \\ 0 & \textcircled{1} & 2 & 0 & 3 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the rows with pivots above (in circles), a basis for $\text{row}(A)$ is:

$$[1\ \ 0\ \ 1\ \ 0\ \ -1], [0\ \ 1\ \ 2\ \ 0\ \ 3], [0\ \ 0\ \ 0\ \ 1\ \ 4]$$

hence $\dim(\text{row}(A)) = 3$

For $\text{col}(A)$: identify columns in A which correspond to pivots in REF (which is B, above). Hence a basis is:

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}, \dim(\text{col}(A)) = 3$$

Theorem:

$$\dim(\text{col}(A)) = \dim(\text{row}(A)) = \text{rank}(A)$$

$$\text{span}(A) = \text{span}(B) \text{ where } B \text{ is REF or RREF of } A$$

For $\text{null}(A)$: we need to compute the parametric form of $A\bar{x} = 0$, which can be obtained from the REF of A or RREF of A.

$$B = \overbrace{\left[\begin{array}{ccccc|c} \textcircled{1} & 0 & 1 & 0 & -1 & 0 \\ 0 & \textcircled{1} & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}^{[x\ \ y\ \ z\ \ w\ \ t]} \Rightarrow \begin{cases} x + z - t = 0 \\ y + 2z + 3t = 0 \\ w + 4t = 0 \end{cases} \Rightarrow \begin{cases} x = -z + t \\ y = -2z - 3t \\ w = -4t \end{cases}$$

z, t are free; x, y, w are leading variables.

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \\ w \\ t \end{bmatrix} = \begin{bmatrix} -z+t \\ -2z-3t \\ z \\ -4t \\ t \end{bmatrix} = z \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Therefore a basis for $\text{null}(A)$ and its dimension are:

$$B = \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}, \dim(\text{null}(A)) = 2$$

Rank theorem

$$n = \#\text{ of columns of A} = \dim\left(\underbrace{\text{row}(A)}_{\substack{\#\text{ of pivots} \\ = \#\text{ of leading variables}}} \right) + \dim\left(\underbrace{\text{null}(A)}_{\substack{\#\text{ of free} \\ \text{variables}}} \right)$$

↑ this looks bad so I'll fix it later

Section 6.3: coordinate systems in $\mathbb{R}^n$

Definition: Given a basis $B = \{v_1, v_2, ..., v_n\}$ of $\mathbb{R}^n$ and a vector $\bar{x} \in \mathbb{R}^n$ then:

$$\bar{x} = a_1 v_1 + a_2 v_2 + ... + a_n v_n$$

for some $a_1, ..., a_n$. The coordinate vector of $\bar{x}$ with respect to the basis B is:

$$[\bar{x}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

We can replace $\mathbb{R}^n$ by a subspace S of $\mathbb{R}^n$. In that case, the number of vectors in B will be equal to $\dim(S)$.

Example: Let $\varepsilon = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$, so $\{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
What is $[\bar{x}]_\varepsilon$ if $\bar{x} = \begin{bmatrix} -1 \\ 0 \\ \pi \end{bmatrix}$?

Solution:

$$\begin{bmatrix} -1 \\ 0 \\ \pi \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore [\bar{x}]_\varepsilon = \begin{bmatrix} -1 \\ 0 \\ \pi \end{bmatrix}$$

Example*:* Let $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ be a basis of a subspace S of $\mathbb{R}^3$. Indeed, $S = \operatorname{span}(B)$.
Is $\bar{x} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}$ in S? If yes, compute $[\bar{x}]_B$.

Solution: Is $\left[\begin{array}{cc|c} 1 & 3 & 7 \\ -1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right]$ consistent?

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ -1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{array}\right]$$

This is consistent, so $\bar{x}$ is in S. To compute $[\bar{x}]_B$, we solve the system:

$$\begin{cases} a + 3b = 7 \\ b = 3 \end{cases} \rightsquigarrow \begin{cases} a = -2 \\ b = 3 \end{cases}$$

Then

$$\begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\rightsquigarrow [\bar{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Example: Given $[x]_B = \frac{1}{5}\begin{bmatrix}13\\6\end{bmatrix}$ and $B = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\}$, compute $\bar{x}$.

Solution: By definition, $[x]_B = \begin{bmatrix}a\\b\end{bmatrix} \Rightarrow \bar{x} = a\begin{bmatrix}2\\1\end{bmatrix} + b\begin{bmatrix}-1\\2\end{bmatrix}$, so $\bar{x} = \frac{13}{5}\begin{bmatrix}2\\1\end{bmatrix} + \frac{6}{5}\begin{bmatrix}-1\\2\end{bmatrix}$.

Lecture 10: change of bases

Section 6.3: coordinate systems in $\mathbb{R}^n$, continued

This is the last topic on the first exam (this Thursday)

Let S be a subspace of $\mathbb{R}^n$ and let $\{B = u_1, ..., u_k\}$ and $C = \{v_1, ..., v_k\}$, two bases of S. For any $\bar{x} \in S$, we have:

1. Using the basis B:

$$\bar{x} = a_1 u_1 + a_2 u_2 + ... + a_k u_k \Leftrightarrow [\bar{x}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_3 \end{bmatrix} \in \mathbb{R}^k$$

2. Using the basis C:

$$\bar{x} = b_1 v_1 + b_2 v_2 + ... + b_k v_k \Leftrightarrow [\bar{x}]_C = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_3 \end{bmatrix} \in \mathbb{R}^k$$

How could we compute $[\bar{x}]_B$ from $[\bar{x}]_C$ (and vice-versa)? We use a *change-of-basis* matrix.

Note: there were some diagrams about mapping matrices between different "worlds" (spaces?) but that would be really hard to pit in here, and also I'm typing this on an iPad right now, so...

What we're looking for is a function that will map a vector in $[\bar{x}]_B$ to $[\bar{x}]_C$. This function will be a change-of-basis matrix $P_{C \leftarrow B}$, which defines the applicable linear transformation. (You can, of course, also find the opposite for going from C to B.)

So how do we compute $P_{C \leftarrow B}$? We use the *Gauss-Jordan method.*

Theorem: Gauss-Jordan method for computing a change-of-basis matrix

$$\left[\underbrace{\left([v_1]_E, ..., [v_k]_E \right)}_{\text{Basis } C} \middle| \underbrace{\left([u_1]_E, ..., [u_k]_E \right)}_{\text{Basis } B} \right] \xrightarrow{\text{Gauss-Jordan elimination}} [I_k \mid P_{C \leftarrow B}]$$

Moreover,

$$P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1}$$

Example: Let $B = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases of $\mathbb{R}^2$. Assume $[\bar{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

1. Compute x

$$[\bar{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \bar{x} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

2. Compute $P_{C \leftarrow B}$

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow P_{C \leftarrow B} = \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix}$$

3. Compute $[\bar{x}]_C$

$$[\bar{x}]_C = P_{C \leftarrow B}[\bar{x}]_B = \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

4. Compute $P_{B \leftarrow C}$

$$P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1} \rightsquigarrow \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix}^{-1} = \frac{1}{-3}\begin{bmatrix} -1 & -3 \\ -2 & -3 \end{bmatrix} = P_{B \leftarrow C}$$

Example: Consider the basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^3$ and let $[\bar{x}]_B = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

1. Find $\bar{x}$

$$x = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + -1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

2. Find $P_{B \leftarrow E}$ (the change-of-coordinate matrix)

$$[\text{basis } B \mid \text{basis } E]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$P_{B \leftarrow E} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

3. If $\bar{y} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, find $[\bar{y}]_B$

$$[\bar{y}]_B = P_{B \leftarrow E}[\bar{y}]_E = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

Lecture 11: vector spaces, bases, and dimension

Firstly, what is a field? It's an algebraic structure with two operations $+$ and $\times$, where we can add, subtract, multiply, and divide by nonzero elements. Examles of fields include:

- the real numbers ($\mathbb{F} = \mathbb{R}$)
- the set of complex numbers ($\mathbb{F} = \mathbb{C}$)
- the set of rational numbers ($\mathbb{F} = \mathbb{Q}$)

However, these things don't form a field:

- polynomials
- integers ($\mathbb{Z}$)
  - this is because if you divide one integer by another, you could end up with a non-integer

Definition of a vector field

A vector space over a field $\mathbb{F}$ is a set $V \neq 0$ on which two operations, called *addition* and *scalar multiplication*, are defined so that: for all $u, v, w \in V$ and $a, b \in \mathbb{F}$, (fill in here)

Example: here are some example of famous vector spaces…

1. $V = \mathbb{R}^n, \mathbb{F} = \mathbb{R}$, with operations:
   - $(u_1, ..., u_n) + (v_1, ..., v_n) = (u_1 + v_1, ...u_n + v_n) \propto (u_1, ..., u_n) = (\propto u_1, ..., \propto u_n)$ for any $\propto \in \mathbb{R} = \mathbb{F}$
   - Basis for $\mathbb{R}^n$: $E = \{e_1, ..., e_n\}$ standard basis; $\dim(\mathbb{R}^n) = \#E = n$
   - $\vec{0} = (0, ..., 0)$ zero vector
2. $V = \mathbb{P}_n = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}\}, \mathbb{F} = \mathbb{R}$, with operations:
   - $(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n) = (a_0 + b_0) = (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$
   - $\alpha(a_0 + a_1 x + \cdots + a_n x^n) = (\alpha a_0) + (\alpha a_1)x + \cdots + (\alpha a_n)x^n$
   - $\dim(\mathbb{P}_n) = \#B = n + 1$
   - In particular, $\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$.
     - $1 + x$ is in $\mathbb{P}_2$, 1 is in $\mathbb{P}_2$, 0 is in $\mathbb{P}_2$, $x^2, \sin(\mathbb{P}_2)$ (not sure what rhose last two mean)
     - Observe that $\mathbb{P}_2 = \operatorname{span}(1, x, x^2)$.
     - Moreover, $B = \{1, x, x^2\}$ is a basis for $\mathbb{P}_2$
       - because $a_0 \cdot 1 + a_1 x + a_2 x^2 = 0 + 0x + 0x^2 \Rightarrow a_0 = a_1 = a_2 = 0$
       - $\dim(\mathbb{P}_2) = \#B = 3$
3. Matrices of size $m \times n$, $\mathbb{F} = \mathbb{R}$ operations given entry-wise.
   - For example,

$$\operatorname{mat}_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta & \chi & \delta \end{bmatrix} = \begin{bmatrix} a + \alpha & b + \beta \\ c + \gamma & d + \delta \end{bmatrix}$$

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

   - In $\operatorname{mat}_{2\times 2}(\mathbb{R})$ the "zero vector" is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   - Also observe the following:

$$\begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + -1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   Then $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $\operatorname{mat}_{2\times 2}(\mathbb{R})$. So $\dim(\operatorname{mat}_{2\times 2}(\mathbb{R})) = \#B = 4$.
   - In general, $\dim(\operatorname{mat}_{m\times n}(\mathbb{R})) = m \cdot n$, and the *standard basis* is

$$B = \left\{ \left(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \cdots \right) \right\}$$

Lecture 12: more on vector spaces

Example: $V = \mathbb{Z}, F = \mathbb{R}$, operations o V as usual (addition of integers ans scalar multiplication between integers and reals). Is $\mathbb{Z}$ a vector space over $\mathbb{R}$?

Solution: Properties 1 through 5 hold because the addition is the usual addition of integers. We need to check properties 6-10. (from definitions document)

$$(6)\ a \in r, u \in V \Rightarrow au \in V = \mathbb{Z}?$$

No, because

$$a = 5.5, u = 3 \in \mathbb{Z}$$

$$a * u = (5.5)(3) = 16.5$$

and 16.5 is not an integer.

Example: $V = \mathbb{R}^2, \mathbb{F} = \mathbb{R}$, and operations:

sum as usual: $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} x+z \\ y+w \end{bmatrix}$

scalar multiplication: $\underbrace{x}_{\mathbb{R}} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbb{R}^2} = \begin{bmatrix} ax \\ 0 \end{bmatrix}$

Solution:

Because addition is as usual, properties 1-5 hold. Now we need to check 6-10:

$$6)\ \alpha \in \mathbb{R}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{R}^2 \Rightarrow \alpha \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ 0 \end{bmatrix} \in \mathbb{R}^2?$$

$$\checkmark \text{ yes}$$

$$7)\ \alpha \in \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{R}^2 \Rightarrow$$

$$\underbrace{\alpha\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix}\right)}_{\alpha(u+v)} = \alpha \begin{bmatrix} x+z \\ y+w \end{bmatrix} = \begin{bmatrix} \alpha(x+z) \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha z \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} x \\ y \end{bmatrix} + \alpha \begin{bmatrix} z \\ w \end{bmatrix}$$

$$\checkmark \text{ yes}$$

$$8)\ \alpha, \beta, \in \mathbb{R}, \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \Rightarrow$$

$$(\alpha + \beta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (\alpha + \beta)(x) \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x + \beta x \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha x \\ 0 \end{bmatrix} + \begin{bmatrix} \beta x \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} x \\ y \end{bmatrix} + \beta(x, y)$$

$$\therefore (a+b)u = au + bu$$

$$\therefore \checkmark \text{ yes}$$

$$9)\ \alpha, \beta \in \mathbb{R}, \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

$$(\alpha\beta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (\alpha\beta)x \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha(\beta x) \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} \beta x \\ 0 \end{bmatrix} = \alpha\left(\beta \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$\therefore (ab)u = a(bu)$$

$$\therefore \checkmark \text{ yes}$$

$$10)\ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \Rightarrow 1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \rightsquigarrow \boxtimes \text{ no}$$

Hence $\mathbb{R}^2$ with the given operations is *not* a vector space over $\mathbb{R}$.

Subspaces, bases, and dimension

Example: Is $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad = 0 \right\}$ a subspace of $\mathrm{Mat}_{2\times2}(\mathbb{R})$?

Soluton: 1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is clearly in S. ✓

2) The sum of two matrices in S is not always itself in S. For example:

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \in S, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \in S$$

but

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$ad = -1 * -1 = 1 \neq 0$$

∴ S is not a subspace.

Example: Is $S = \{A \in \text{Mat}_{2\times 2}(\mathbb{R}) : A^T = A\}$. Is S a subspace of $\text{Mat}_{2\times 2}$? If so, find a basis for S. (S is the set of all symmetric matrices of size 2x2.)

Solution:

Firstly,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$$
$$\therefore A^T = A \checkmark$$

Next, assume $A, B \in S$. Then

$$(A + B)^T = A^T + B^T = A + B \therefore \checkmark$$

(because $A = A^T$ and $B = B^T$)

Finally, for $a \in \mathbb{R}, A \in S$:

$$(aA)^T = aA^T = aA \therefore \checkmark$$

(because $A = A^T$ since A is in S)

Therefore, since all three criteria hold, S is a subspace of $\text{Mat}_{2\times 2}(\mathbb{R})$.

Ok now finding a basis for S

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$A = A^T \Leftrightarrow \begin{cases} a = a \\ b = c \\ d = d \end{cases}$$

So the matrices in S are of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$. Hence,

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

And that means that a basis for S is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Exam 1 review lecture

Question 1: Consider the system $A\bar{x} = b$ with the augmented matrix

$$[A \mid b] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Which of the following $\bar{x}$ are solutions of the system?
a) $\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ b) $\bar{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 5 \\ 2 \end{bmatrix}$, c) $\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, d) $\bar{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, e) there are no solutions

Solution: (b) is correct. Just check whether the vectors are solutions, and if there is a solution, that's the right answer since there is only one soluton *(jack note to self: add more here)*

Question 2: If A is a 2x2 matrix such that

$$\left(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}^T\right)^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

then what is the matrix A?

Solution:

- Apply the following properties of the transpose:
  1. $(A+B)^T = A^T + B^T$
  2. $(kA)^T = kA^t$
  3. $\left(A^T\right)^T = A$
- So then we can do

$$\left(2A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}^T\right)^T = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$2A - 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$A = \frac{1}{2}\left(\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}\right)$$

$$A = \frac{1}{2}\begin{bmatrix} 5 & 9 \\ -4 & 5 \end{bmatrix}$$

Question 3: which of the following are not satisfied for all matrices A, B?

1. $\left(A^T\right)^T \longrightarrow$ true
2. $(A+B)^2 = A^2 + 2AB + B^2 \longrightarrow$ false
   - This is not valid because $AB \neq BA$, so we can't simplify to $2AB$
3. $A(B+C) = AB + BC \longrightarrow$ true
4. $(A-B)(A+B) = A^2 - B^2 \longrightarrow$ false

- there are more here which I missed

Question 4: A map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is a counterclockwise rotation about the origin through $\frac{\pi}{2}$ radians is a linear transformation. What is the standard matrix $[T]$ of T?

Solution:

- Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has associated a standard matrix $[T]$ for which $T(\bar{x}) = [T]\bar{x}$. Indeed:

$$[T] = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

So

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ x \end{bmatrix} \rightsquigarrow [T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

A second way to obtain $[T]$ is by computing T on the standard basis of $\mathbb{R}^2$.

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, T(e_2) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Now, solving the problem: we're rotating a vector by 90 degrees (sorry it's hard to draw a graph in here). So then

$$T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$$

$$T(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So

$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Question 5: The maps $T : \mathbb{R}^4 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ given below define linear transformations:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix}, S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

For this problem, a) compute the standard matrix $C = [S \circ T]$, and b) give a formula for $SoT : \mathbb{R}^4 \to \mathbb{R}^3$, like the transformations above.

Solution: With $T : \mathbb{R}^n \to \mathbb{R}^m$, $S : \mathbb{R}^m \to \mathbb{R}^k$ as linear transformations, then $S \circ T : \mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation and

$$[S \circ T] = [S] \cdot [T]$$

So now for part (a) we can calculate:

$$[S \circ T] = [S] \cdot [T]$$

$$[S \circ T] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[S \circ T] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

And then for part (b), in which we're doing $S \circ T : \mathbb{R}^4 \to \mathbb{R}^3$:

$$(S \circ T)\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = [S \circ T]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + 2x_2 + x_3 \\ x_2 + 2x_3 + x_4 \end{bmatrix}$$

Question 6: Let A be a 5x8 matrix such that $\dim(\text{null}(A)) = 3$. Compute the dimension of $\text{null}(A)$.

Recall the rank theorem, which says that

$$\#\text{ columns of A} = \begin{cases} \text{rank}(A) + \dim(\text{null}(A)) \\ \dim(\text{col}(A)) + \dim(\text{null}(A)) \\ \dim(\text{row}(A)) + \dim(\text{null}(A)) \end{cases}$$

(All of the things on the right are equivalent statements.)

Ok so in the context of Q6,

$$8 = \dim(\text{col}(A)) + 3 \Rightarrow \dim(\text{col}(A)) = 5$$

$$\underbrace{A}_{5\times 8} \xrightarrow[]{\substack{\text{columns} \\ \text{become rows}}} \underbrace{A^T}_{8\times 5}$$

$$\#\text{ columns of } A^T = \dim(\text{row}(A^T)) + \dim()$$

uhhhh there's more here which i missed

Lecture 14: linear transformations (part 1)

Definition: let V, W be vector spaces over a field $\mathbb{F}$.

A function $T : V \to W$ is a linear transformaton if:

1. $T(u+v) = T(u) + T(v)$ ("open sums")
2. $T(ku) = kT(u)$ ("take out the scalars")

Example: $T : P_2(\mathbb{R}) \to \text{Mat}_{2\times 2}(\mathbb{R})$, so $T(a + bx + cx^2) \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that T is a linear transformation.

Solution:

1. "open sums": given

$$\begin{cases} u = a + bx + cx^2 \\ v = \alpha + \beta x + \gamma x^2 \end{cases} \in P_2(\mathbb{R})$$

so

$$T(u+v) = T((a+\alpha) + (b+\beta)x + (c+\gamma)x^2)$$

$$= \begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$$

$$= T(a + bx + cx^2) + T(\alpha + \beta x + \gamma x^2)$$

$$= T(u) + T(v)$$

hence

$$T(u+v) = T(u) + T(v) \checkmark$$

2. "take out scalars"

Let $k \in \mathbb{R}, u = a + bx + cx^2 \in P_2(\mathbb{R})$

$$T(ku) = T(k(a + bx + cx^2)) = T((ka) + (kb)x + (kc)x^2)$$

$$= \begin{bmatrix} ka & kb \\ kc & 0 \end{bmatrix} = k \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = kT(u) \checkmark$$

Hence, T is a linear transformation.

Example: is $T : \text{Mat}_{m\times n}(\mathbb{R}) \to \text{Mat}_{n\times n}(\mathbb{R}), T(A) = A^T$ a linear transformation?

1. "open sums"

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

2. "take out scalars"

$$T(kA) = (kA)^T = kA^T = kT(A)$$

So T is a linear transformation.

Example: We defime two vector spaces

$$D = \{f : \mathbb{R} \to \mathbb{R} \mid f' \text{ exists}\}$$

$$F = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a function}\}$$

We now define $D \to F$ as $D(f) = f'$. Prove that D is a linear transformstion. (This transformation is called the *differential operator*.)

1. "open sums" Let $f, g \in D$. Then

   $$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$$

   Yep, it open sums!
2. "take out the scalars" Let $k \in \mathbb{R}$, $f \in D$. Then

   $$D(kf) = (kf)' = kf' = kD(f)$$

   You can also take out the scalars!

Properties of linear transformations

Let $T : V \to W$ be a linear transformation.

1. $T(0_V) = 0_W$
2. $T(u - v) = T(u) - T(v)$
3. The action of T on every vector in V is completely determined by the action of T on any basis B of V.
   - If $B = \{v_1, ..., v_n\}$, then in order to kmow $T(v)$, you only need to know $T(v_1), ..., T(v_n)$

Example: suppose $T : \mathbb{R}^2 \to \mathbb{P}_2(\mathbb{R})$ is a linear transformaton such that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2 - 3x + x^2$$

$$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = 1 - x^2$$

Compute $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$.

1. Computing $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)$:

   Notice that $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$ (which is the domain of T). Then we can do

$$\begin{bmatrix}-1\\2\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}2\\3\end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c}1 & 2 & -1\\1 & 3 & 2\end{array}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{array}{cc|c}1 & 0 & -7\\0 & 1 & 3\end{array}\right] \Rightarrow \begin{cases}x = -7\\y = 3\end{cases}$$

   Now we can check that

$$\begin{bmatrix}-1\\2\end{bmatrix} = -7\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}2\\3\end{bmatrix}$$

   and that checks out so we're all good! So now we can do this

$$T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = T\left(\underbrace{-7\begin{bmatrix}1\\1\end{bmatrix}}_{u} + \underbrace{3\begin{bmatrix}2\\3\end{bmatrix}}_{v}\right) = T\left(\underbrace{-7\begin{bmatrix}1\\1\end{bmatrix}}_{ku}\right) + T\left(\underbrace{3\begin{bmatrix}2\\3\end{bmatrix}}_{kv}\right) = -7T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + 3T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$$

$$= -7(2 - 3x + x^2) + 3(1 - x^2) = -11 + 21x - 10x^2$$

2. Computing $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$

   Similarly,

$$\begin{bmatrix}a\\b\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}2\\3\end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c}1 & 2 & a\\1 & 3 & b\end{array}\right] \rightsquigarrow \begin{cases}x = 3a - 2b\\y = b - a\end{cases}$$

   Then

$$\begin{bmatrix}a\\b\end{bmatrix} = (3a - 2b)\begin{bmatrix}1\\1\end{bmatrix} + (b - a)\begin{bmatrix}2\\3\end{bmatrix}$$

   so

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = T\left((3a - 2b)\begin{bmatrix}1\\1\end{bmatrix} + (b - a)\begin{bmatrix}2\\3\end{bmatrix}\right)$$

$$= (3a - 2b)T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + (b - a)T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = (5a - 3b) + (-9a + 6b)x + (4a - 3b)x^2$$

Lecture 15: kernel, rank, and isomorphisms

Let $T : V \to W$ be a linear transformation.

Definition of the kernel of T:

$$\ker(T) = \{v \in V : T(v) = 0_w\}$$

(Vectors in V that are mapped to zero in W)

Definition of the range of T:

$$\text{range}(T) = \{T(v) : v \in V\}$$

(The image of the function T)

Theorem: $\ker(T)$ is a subspace of V

Definiton: the nullity of T is defined as

$$\text{nullity}(T) = \dim(\ker(T))$$

Theorem: the range of T is a subvector space of W.

Definiton: the rank of T is defined as

$$\text{rank}(T) = \dim(\text{range}(T))$$

Theorem: Rank Theorem

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Theorem
For $T : \mathbb{R}^m \to \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$:
- $\text{range}(T) = \text{col}(A)$
- $\text{rank}(T) = \text{rank}(A) = \dim(\text{col}(A))$

Example: given $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find...

1. $\ker(T)$

$$\ker(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \Leftrightarrow \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving the homogenous system:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix} \rightsquigarrow \ker(T) = \operatorname{span}\left(\begin{bmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ 1 \end{bmatrix}\right)$$

2. $\operatorname{nullity}(T)$

$\operatorname{nullity}(T) = 1$ because $\dim(\ker(T)) = 1$

3. $\operatorname{range}(T)$

By definition,

$$\operatorname{range}(T) = \left\{ T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \right\}$$

So now we look at the condition that defines the range and work with that condition.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \operatorname{col}(A)$$

Ok so since this equals column space, now we can just work to find a basis for that:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} \end{bmatrix}$$

so

$$\operatorname{col}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$$

4. $\operatorname{rank}(T)$
$\operatorname{rank}(T) = 2$ because $\dim(\operatorname{range}(T)) = 2$

Example: given $T : P_2(\mathbb{R}) \to P_3(\mathbb{R}), T(p(x)) = x^2 p'(x)$...

0. Verify that this is a linear transformation
1. Find a basis for range(T) and find rank(T)
2. Find a basis for ker(T) and find nullity(T)

Solution:

- For part 1:

$$\text{range}(T) = \{T(p(x)) : p(x) \in \mathbb{P}_2(\mathbb{R})\}$$

$$T(p(x) = x^2 * p(x) = x^2(a + bx + cx^2)) = x^2(b + 2cx) = bx^2 + 2cx^3$$

so

$$\text{range}(T) = \text{span}(x^2, x^3)$$

and that means a basis for range(T) is

$$B = \{x^2, x^3\}$$

Finally,

$$\text{rank}(T) = 2$$

(there are two linearly independent vectors in the basis)

- For part 2:

$$\ker(T) = \{p(x) \in \mathbb{P}_2(x) : T(p(x)) = 0\}$$

$$T(p(x)) = x^2 * p'(x) = bx^2 + 2cx^3 = 0 + 0x + 0x^2 + 0x^3$$

Comparing the coefficients in the last two things above tells you that

$$\begin{cases} b = 0 \\ 2c = 0 \end{cases} \rightsquigarrow \begin{cases} b = 0 \\ c = 0 \end{cases}$$

so $p(x) = a + bx + cx^2$ is in ker(T) if and only if $b = 0$ and $c = 0$. This means

$$p(x) = a$$

i.e., only constant polynomials are in the kernel of T. And now we can say that

$$\ker(T) = \text{span}(1) = \mathbb{R}$$

and

$$\text{nullity}(T) = 1$$

Lecture 16: isomorphisms and matrices of transformation

Recall that a function $f : A \to B$ is one-to-one if:

- For two inputs $a \neq b$, $f(a) \neq f(b)$
  - (for distinct inputs, there are distinct outputs)
- Or equivalently, $f(a) = f(b) \Rightarrow a = b$
  - ($a = b$ iff $f(a) = f(b)$)

A function $f : A \to B$ is onto if:

- $\{f(a) : a \in A\} = B$
  - (the image of $f(a)$ onto A is B) [← check this]

Example: $f : \mathbb{R} \to \mathbb{R}$ given by

1. $f(x) = x^2$
   - Not one-to-one, because $f(1) = f(-1)$
   - Not onto, because $\text{range}(f) \neq \mathbb{R}$
2. $f(x) = x + 1$
   - One-to-one and onto
   - In this case, f is also a bijection

Definition of isomorphisms: Let V, W be vector spaces over the same field $\mathbb{F}$ and $T : V \to W$ be a linear transformation. We say that T is an *isomorphism* if T is one-to-one and onto.

Theorem: Let $T : V \to W$ be a linear transformation. Then:

1. T is one-to-one if and only if $\ker(T) = \{0_W\}$
2. T is onto if and only if $\text{range}(T) = W$
   - or if and only if $\text{rank}(T) = \dim(W)$
     - (this is because $\text{rank}(T) = \dim(\text{range}(T))$)
3. T is an isomorphism if and only if $\ker(T) = \{0\}$ and $\text{rank}(T) = \dim(W)$

Example: Determine if the given transformation is one-to-one, onto, or an isomorphism.

1. $T : P_2(\mathbb{R}) \to P_2(\mathbb{R}), T(p(x)) = p(0)$

Solution:

a) Is T one-to-one?

$$\ker(T) = \{p(x) : T(p(x)) = 0\}$$

If $p(x) = a + bx + cx^2$, then $p(0)$ = a = 0. This means that $p(x) \in \ker(T)$ iff $p(x) = bx + cx^2$
So for example, $p(x) = x + x^2$ satisfies $p(0) = 0$. This means that $p(x) \in \ker(T) \Rightarrow \ker(T) \approx \{0\}$. This violates the first condition of the above theorem. Hence, T is not one-to-one.

b) Is T onto?

$$\text{range}(T) = \{T(p(x)) : p(x) \in P_2(\mathbb{R})\}$$

$$p(x) = a + bx + cx^2 \Rightarrow T(p(x)) = p(0) = a + b(0) + c(0)^2 = a$$

Hence, $\text{range}(T) = \text{span}(1) \neq P_2(\mathbb{R}))$. So T is not onto.

c) Is T an isomorphism?
Clearly not, since to be an ismomorphism a linear transformaion must be both one-to-one and onto… and T is neither.

Example: $T : \mathbb{R}^2 \to P_1(\mathbb{R}), T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x$

a) Is T one-to-one?

$$\ker(T) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 : T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = 0 + 0x \right\}$$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + (a+b)x = 0 + 0x$$

Comparing coefficents,

$$\begin{cases} a = 0 \\ a + b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases} \Rightarrow \ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

So T is one-to-one. ✓

b) Is T onto?
By the rank theorem,

$$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(\mathbb{R}^2) = 2$$

$$0 + \text{rank}(T) = 2$$

$$\Rightarrow \text{rank}(T) = 2 = \dim(P_1(\mathbb{R}))$$

So T is onto. ✓

c) Is T an isomorphism?
Yes, because T is both one-to-one and onto. ✓

In this case, we say that $\mathbb{R}^2$ is isomorphic to $P_1(\mathbb{R})$, and this relationship is written in the following way:

$$\mathbb{R}^2 \cong P_1(\mathbb{R})$$

To be isomorphic means that "they look the same from a math perspecitve".

Example: $T : \mathbb{R}^4 \to P_4(\mathbb{R}), T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = a + bx + cx^2 + dx^3$

- This is clearly one-to-one, since the only way to get the zero vector
- The image is only going to produce polynomials of degree 3, but $P_4(\mathbb{R})$ includes polynomials of degree 4, and you can never get a polynomial of degree 4 from T.
- Therefore this transformation is one-to-one but not onto.

Also:

$$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(\mathbb{R}^4)$$

$$0 + \dim(\text{range}(T)) = 4$$

But $\dim(P_4(\mathbb{R})) = 5$, so

$$\dim(\text{range}(T)) < \dim(P_4(\mathbb{R}))$$

Matrices of linear transformations

Recall that for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we can compute the standard matrix of T:

$$[T]_{E_m \leftarrow E_n} = [T(e_1) \ \dots \ T(e_n)]$$

We generalize for two vector spaces.

Theorem: Let $T : V \to W$ be a linear transformation. B is a basis for V, and C is a basis for W. The the matrix of T with respect to B and C is

$$[T]_{C \leftarrow B} = \left[[T(v_1)]_C \ \ [T(v_2)]_C \ \ \dots [T(v_n)]_C \right]$$

$$B = \{v_1, \dots, v_n\}$$

Lecture 17: coordinate vectors

Matrices of linear transformations, continued

Example: let $D : P_3 \to P_2$ be the differential operator, $B = \{1, x, x^2, x^3\}$, and $C = \{1, x, x^2\}$.
a) compute $[D]_{C \leftarrow B}$, and
b) use part (a) to compute $D(5 - x + 2x^3)$.

Solution:

For part (a):

$$[D]_{C \leftarrow B} = \begin{bmatrix} [D(1)]_C & [D(x)]_C & [D(x^2)]_C & [D(x^3)]_C \end{bmatrix}$$

$$D(1) = 0 = 0 * 1 + 0x + 0x^2 \rightsquigarrow [D(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x) = 1 = 1 * 1 + 0x + 0x^2 \rightsquigarrow [D(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x^2) = 2x = 0 * 1 + 2x + 0x^2 \rightsquigarrow [D(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$D(x^3) = 3x^2 = 0 * 1 + 0x + 3x^2 \rightsquigarrow [D(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Hence,

$$[D]_{C \leftarrow B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Now for part (b):

$$p = 5 - x + 2x^3$$

$$[p]_B = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

so

$$[D(p)]_C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

This is the coordinate of the derivative of the polynomial in the basis C. So then

$$D(p) = -1 * 1 + 0x + 6x^2 = -1 + 6x^2$$

Matrices of compositions of transformations

If you have U, V, and W bridged by transformations T and S, you can skip the middle step and go straight from U to W with the composite of S and T: $(S \circ T)(u)$

Theorem: $[S \circ T] = [S]_{D \leftarrow C}[T]_{C \leftarrow B}$

Example: $T : P_1 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ are given by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$ and $S\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a-2b \\ 2a-b \end{bmatrix}$. The bases are $B = \{1, 1+x\}$, $C = \{e_1, e_2\}$, and $D = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, respectively.
a) Compute $[S \circ T]$, and
b) Compute $(S \circ T)(2 + x)$ using (a).

Solution: For part (a):

$$[S \circ T]_{D \leftarrow B} = [S]_{D \leftarrow C} [T]_{C \leftarrow B}$$

So first let's find

$$[T]_{C \leftarrow B} = \begin{bmatrix} [T(1)]_C & [T(1+x)]_C \end{bmatrix}$$

$$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1e_1 + 1e_2$$

$$T(1+x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1e_1 + 2e_2$$

Hence

$$[T]_{C \leftarrow B} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Each column comes from the coefficients of e_n in the preceding two lines.

Ok next let's find

$$[S]_{D \leftarrow C} = \begin{bmatrix} [S(e_1)]_D & [S(e_2)]_D \end{bmatrix}$$

$$S(e_1) = S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S(e_2) = S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence (putting the coefficients from above into the columns, like we did before)

$$[S]_{D \leftarrow C} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$$

Finally, to answer (a):

$$[S \circ T]_{D \leftarrow B} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}$$

Alright now for part (b):

$$p = 2 + x = 1(1) + 1(1 + x) \rightsquigarrow [p]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so

$$[S \circ T]_{D \leftarrow B} [p]_B = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} = [(S \circ T)(p)]_D$$

$D = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, so putting it back into standard basis:

$$(S \circ T)(p) = -5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Change of basis in vector spaces

V vector space over $\mathbb{F}$, $B = \{v_1, ..., v_n\}$, C bases for V.
Any vector $v \in V$ can be written in terms of B or C.

Question: $[v]_C \overset{?}{\approx} [v]_B$?

Answer: By the change of basis matrix $P_{C \leftarrow B} = [I]_{C \leftarrow B}$, $I : V \to V$ is the identity map, so $I(v) = v$.

Example: $V = P_2, B = \{1, x, x^2\}, C = \{1 + x, x + x^2, 1 + x^2\}$. Find $P_{B \leftarrow C}$

Solution:

$$P_{B \leftarrow C} = \begin{bmatrix} [1+x]_B & [x+x^2]_B & [1+x^2]_B \end{bmatrix}$$

$$P_{B \leftarrow C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Lecture 18: determinants

Determinants are defined for $(n \times n)$ square matrices only and, in this course, they will be computed using cofactor expansion along any row or any column.

Definition: Given a square matrix A, we define the

- (i, j)-minor as $\det\left(\tilde{A}\right)$ where $\tilde{A} =$ remove row i and column j of A
- (i, j)-cofactor as $(-1)^{i+j} \det\left(\tilde{A}\right)$

Example: Given

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- the $(2, 2)$-minor is

$$\det\left(\tilde{A}\right) = \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\right) = 1 * 2 - 0 * 2 = 2$$

- the $(2, 2)$-cofactor of A is:

$$(-1)^{2+2} \det\left(\tilde{A}\right) = 1 \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\right) = 1 * 2 = 2$$

Example: Compute $\det(A)$ where $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Solution 1: We will do cofactor expansion along row 2.

$$\det(A) = 3 * (2, 1)\text{-cofactor of } A + 1 * (2, 2)\text{-cofactor of } A + 0 * (2, 3)\text{-cofactor of } A$$

$$\det(A) = 3 * (-1)^{2+1} \det\left(\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}\right) + 1 * (-1)^{2+2} \det\left(\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\right) + 0 * (-1)^{2+3} \det\left(\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\right)$$

$$\det(A) = 3(-1)(-2) + 1(1)(2) + 0$$

$$\det(A) = 8$$

As you can see, the smartest way to go about doing cofactor expansion is to pick the row or column with the most zeros, since every time there's a zero, you don't have to compute the determinant and everything.

Solution 2: Now we'll pick the row or column with the most zeros, so we'll do cofactor expansions along row 3.

$$\det(A) = 2 * (3, 3)\text{-cofactor of } A$$

$$\det(A) = 2(-1)^{3+3} \det\left(\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\right)$$

$$\det(A) = 2(1)(1 - (-3)) = 2 * 4$$

$$\det(A) = 8$$

Example: Compute $\det(A)$ where $A = \begin{bmatrix} 1 & 4 & 7 & 700 \\ 0 & -1 & 100 & 600 \\ 0 & 0 & 6 & 200 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution: This is an upper triangular matrix, so there are a bunch of zero entries. We can easily compute the determinant using recursive cofactor expansion a long the first column:

$$\det(A) = (1)(-1)(6)(3) = 18$$

This turns out to just entail taking the product of the elements in the main diagonal (top left to bottom right). Indeed, this holds for all upper and all lower triangular matrices:

$$\det(A) = \begin{matrix} \text{product of elements in main diagonal} \\ \text{for upper triangular matrices} \end{matrix}$$

Remark: $\det(I_3) = 1$; in general, $\det(I_n) = 1$.

The determinant is a function $\det : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$. We would like to study the properties of this function.

Properties:

1. $\det(AB) = \det(A)\det(B)$
   - If $\det(A) = 4$ and $\det(B) = -3$, then $\det(AB) = (4)(-3) = -12$
2. A is invertible $\Leftrightarrow \det(A) \neq 0$
3. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
   - This is because $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$

Example: is $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ invertible?

Solution: from the previous example, we know that $\det(A) = 8 \neq 0$; $\therefore A$ is invertible

Example: Given $\det(A) = 3, \det(B) = -1$, compute $\det(A^{-1}B^2)$

Solution:

$$\det(A^{-1}B^2) = \det(A^{-1})\det(B^2) = \det(A^{-1})\det(B)\det(B)$$

.

$$= \frac{1}{3}(-1)^2 = \frac{1}{3}$$

Lecture 19: more on determinants, Cramer's rule, eigenvalues

Note: eigenvalues will definitely not be on this upcoming exam (it was unclear whether Cramer's rule will be)

More on determinants

From last time…

1. If A is a $n \times n$ (square) matrix, then $\det(A) = |A|$ is defined and computed using cofactor expansions along any row or column
   - $|A|$ looks like an absolute value, but that's actually the shorthand notation for the determinant
   - Pick the row/column with the most zeros because it's easiest
   - Also, for upper and lower triangular matrices, the determinant is just the product of the main diagonal
2. Properties of determinants (properties 1-3 from last time; properties 4+ are new):
   1. $\det(AB) = \det(A)\det(B)$
   2. A is invertible $\Leftrightarrow \det(A) \neq 0$
   3. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$
   4. $\det(A^T) = \det(A)$
   5. If A has a row or column of zeros, then $\det(A) = 0$
   6. $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $\det(A) = -\det(B)$
   7. $A \xrightarrow{kR_i} B$ then $\det(A) = \frac{1}{k}\det(B)$
   8. $A \xrightarrow{R_i \leftarrow R_i + kR_j} B$ then $\det(A) = \det(B)$

***Examples** of the above properties:*

For property 6:

$$\begin{vmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

For property 7:

$$\begin{vmatrix} 2 & 4 & 6 \\ 3 & 1 & -1 \\ 0 & 3 & 4 \end{vmatrix} \overset{\frac{1}{2}R_1}{=} \frac{1}{\frac{1}{2}} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \\ 0 & 3 & 4 \end{vmatrix}$$

Example: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 3 & 1 & -1 \\ 0 & 3 & 4 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 4 & 6 \\ 3 & 1 & -1 \\ 0 & 3 & 4 \end{vmatrix} \underset{\text{prop 7}}{\overset{\frac{1}{2}R_1}{=}} 2 \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \\ 0 & 3 & 4 \end{vmatrix} \underset{\text{prop 8}}{\overset{R_2 \leftarrow R_2 - 3R_1}{=}} 2 \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & 3 & 4 \end{bmatrix}$$

oops I missed the rest of this

Example: Compute $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & \pi & e & e^2 \\ \sin(2) & \sin(3) & 0 & 1 \\ -2 & 2 & -4 & 6 \end{vmatrix}$.

Solution: $\det(\text{above matrix}) = 0$ because the 4th row is just a multiple of the first

Cramer's rule

Let A be an $n \times n$ invertible matrix and $\vec{b} \in \mathbb{R}^n$. Then the system $A\vec{x} = \vec{b}$ has solution

$$x_i = \frac{\det\left(A_i\left(\vec{b}\right)\right)}{\det(A)}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $A_i\left(\vec{b}\right)$ is the matrix resulting from A by replacing column i of A by $\vec{b}$.

Cramer's rule is useful for when you have a giant matrix but you only would like to know one or a few of the solutions. (Otherwise, Gaussian/Gauss-Jordan elimination is usually less computationally expensive.)

Example: Solve $\begin{cases} x_1 + 2x_2 = 2 \\ -x_1 + 4x_2 = 1 \end{cases}$ using Cramer's rule.

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \ \det(A) = 4 - (-2) = 6 \neq 0$$

$$x_1 = \frac{\det\left(A_1\left(\vec{b}\right)\right)}{\det(A)} = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}}{6} = \frac{8-2}{6} = 1$$

$$x_2 = \frac{\det\left(A_2\left(\vec{b}\right)\right)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}{6} = \frac{1-(-2)}{6} = \frac{1}{2}$$

The adjoint matrix

Definition: Let A be an $n \times n$ matrix. We define the adjoint matrix as

$$\text{adj}(A) = [(i,j)\text{-cofactors of } A]^T$$

which is also an $n \times n$ matrix.

Theroem: if A is invertible, then $A^{-1} = \frac{1}{\det(A)} * \text{adj}(A)$

Example: Given $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, find $\operatorname{adj}(A)$.

Solution:

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

where $C_{ij} = (i, j)$-cofactor of A

$$C_{11} = (-1)^{1+1}|4| = 4$$

$$C_{12} = (-1)^{1+2}|-1| = 1$$

$$C_{21} = (-1)^{2+1}|2| = -2$$

$$C_{22} = (-1)^{2+2}|1| = 1$$

then

$$\operatorname{adj}(A) = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}^T$$

so

$$A^{-1} = \frac{1}{\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix}} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}^T$$

$$= \frac{1}{6} \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues

Motivation: compute $\det\left(\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} 1 - x & -1 \\ 2 & 3 - x \end{bmatrix}\right) = (1 - x)(3 - x) + 2 = 3 - 4x + x^2 + 2 = x^2 - 4x + 5$$

which is a polynomial with entries in $\mathbb{R}$.

What do the roots of the polynomial $x^2 - 4x + 5$ tell us about the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$?

Lecture 20: eigenvalues, eigenvectors, and similarity

Definition: eigenvalue and eigenvector

Given an $n \times n$ matrix A, $\lambda \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$:

1. λ is an eigenvalue of A if there exists a nonzero vector $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} = \lambda\bar{x}$
   - i.e.
2. The vector $\bar{x}$ above is called an eigenvector

Example: given $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \lambda = 4, \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$

Then λ is an eigenvalue of A and $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$ because

$$A\bar{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda\bar{x}$$

Finding eigenvalues and eigenvectors

How do we find the eigenvalues of a matrix and the corresponding eigenvectors?

Theorem: given an $n \times n$ matrix A:

1. The eigenvalues are zeros of the charactaristic polynomial, which is

$$\mathrm{char}_A(x) = \det(A - \lambda I_n)$$

1. For an eigenvalue λ, the eigenvectors corresponding to λ are the nonzero vectors in the subspace E_λ of $\mathbb{R}^n$ (which is called the *eigenspace corresponding to* λ):

$$E_\lambda = \mathrm{null}(A - \lambda I_n)$$

Example: given $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$...

The eigenvalues are

$$\text{char}_A(x) = \left| \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 3-x & 1 \\ 1 & 3-x \end{bmatrix} \right| = (3-x)^2 - 1$$

$$= x^2 - 6x + 9 - 1 = x^2 - 6x + 8$$

$$= (x-4)(x-2)$$

So the eigenvalues are $\lambda = 4$ and $\lambda = 2$

Now to find the eigenvectors!

For $\lambda = 4$:

$$A - \lambda I_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

...now take the null space of that...

$$\text{null}(A - \lambda I_n) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

so

$$-x + y = 0 \rightsquigarrow y = x \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_{\lambda=4} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

For $\lambda = 2$:

$$A = 2I_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x + y = 0 \rightsquigarrow y = -x \rightsquigarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix}, x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$E_{\lambda=2} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

Similarity and diagonalization

Similarity

Definition: Two matrices A and B are called similar (notated as $A \sim B$) if there exists an invertible matrix P such that $B = P^{-1}AP$.

Example: Prove that $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

Take

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Diagonalization

Definition: A matrix A is diagonalizable is $A \sim D$ where D is a diagonal matrix.

Note: Similarity is defined for any two matrices A and B (no matter the form), while diagonalization means $A \sim D$ where D is diagonal (so it's like diagonalization is a special case of similarity, it seems like?)

Example: Diagonalize $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

This isn't diagonal, but it is similar (I think?).

Theorem: Given that $A \sim B$,

1. $\det(A) = \det(B)$
2. $\text{char}_A(x) = \text{char}_B(x)$
3. A is invertible $\Leftrightarrow$ B is invertible
4. A and B have the same eigenvalues
   - A and B do not necessarily have the same eigenvectors (! check about this)

It's important to remember that if A and B have the same eigenvalues and eigenvectors, they're neither necessarily similar nor equal — i.e., the above criteria are only one-way definitions. (This could be an exam question!)

Also:

- A is invertible $\Leftrightarrow$ 0 is not an eigenvalue

Note: Similarity is an "equivalence relation": you can pick a representative form each equivalence class (group of matrices that are similar), and when possible we'll pick diagonal matrices. But not all classes necessarily have diagonal matrices, and for those classes, the next representatives are Jordan blocks (but those aren't covered in this class)

So when can we pick a diagonal matrix as a representative?

Theorem: A matrix A is diagonalizable if and only if the algebraic multiplicity of $\lambda =$ the geometric multiplicity of λ for all eigenvalues λ of A.

- The algebraic multplicity of λ is the maximum power of $x - \lambda$ that divides $\text{char}_A(x)$
- The geometric multiplicity of λ is $\dim(E_\lambda)$

Example: given $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \operatorname{char}_A(x) = (x-4)(x-2)$

Also, we already computed that

$\lambda = 4 \to E_{\lambda=4} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$\lambda = 1 \to E_{\lambda=1} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

λ	algebraic multiplicity	geometric multiplicity
4	1	1
2	1	1

We notice that the algebraic multiplicity = the geometric multiplicity for all λ. Hence A is diagonalizable

$A \sim \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Exam 2 review lecture

List of topics

- § 6.1: vector spaces and subspaces
- § 6.2: linear independence, bases, and dimension
- § 6.3: change of basis matrices $P_{C \leftarrow B}$
- § 6.4: linear transformations
- § 6.5: $\ker(T), \operatorname{range}(T), \operatorname{nullity}(T), \operatorname{rank}(T)$
- § 6.6: matrix of a linear transformation $[T]_{C \leftarrow B}$
- § 4.2: determinants (cofactor expansion, properties of determinants, adjoint matrices, Cramer's rule, computing determinants using REF)
- No eigenvalues or eigenvectors!

Problem 1: Which of the following is a subspace of $\operatorname{Mat}_{2\times 2}(\mathbb{R})$?

1. $W = \{A \in \operatorname{Mat}_{2\times 2}(\mathbb{R}) : \det(A) = 1\}$

$\longrightarrow$ No, because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W$

2. $W = \{A \in \operatorname{Mat}_{2\times 2}(\mathbb{R}) : \det(A) \geq 0\}$

$\longrightarrow$ No, because $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in W$ and $B = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \in W$ but $A + B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \notin W$

3. $W = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R}\right\}$

$\longrightarrow$ Yes, because $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $W = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$

4. $W = \left\{\begin{bmatrix} a & b \\ b & 2a \end{bmatrix} : a, b \in \mathbb{R}\right\}$

$\longrightarrow$ Yes, because $\begin{bmatrix} a & b \\ b & 2a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so $W = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$

Problem 2: Which of the following statements is false?

1. Any set with 6 linearly independnet matrices of $\mathrm{Mat}_{3\times 2}(\mathbb{R})$ is a basis of $\mathrm{Mat}_{3\times 2}(\mathbb{R})$

$\longrightarrow$ True, because $\dim(\mathrm{Mat}_{3\times 2}(\mathbb{R}) = 3 * 2 = 6)$: the dimension of the vector space $\mathrm{Mat}_{3\times 2}(\mathbb{R})$ is 6, so any set of 6 linearly independent matrices will form a basis

2. P_3 can be spanned by 5 vectors

$\longrightarrow$ True, because they don't need to be linearly independent
(For example, $P_3 = \mathrm{span}(1, x, x^2, x^3, 1 + x)$)

3. Any set with 3 vectors in $\mathbb{R}^3$ is linearly independent

$\longrightarrow$ False, because not all vectors in $\mathbb{R}^3$ are necessarily linearly independent: for example, the vectors in the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right\}$ are not linearly independent.

4. A basis of P_3 can't have more than 3 linearly independent vectors

$\longrightarrow$ False, because $\dim(P_3) = 4$

5. There exists a basis of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ with 6 vectors.

$\longrightarrow$ False, because $\dim(\mathrm{Mat}_{3\times 2}(\mathbb{R})) = 4$ (it's always exactly 4)

Problem 3: Let $T : \mathbb{R}^2 \to P_2(\mathbb{R})$ be a linear transformation such that $T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = 2x$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 1 - x$. Compute $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

Represent $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as a combination of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} -1 & 2 & 2 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{5}{3} \end{array}\right] \rightsquigarrow \begin{cases} a = \frac{4}{3} \\ b = \frac{5}{3} \end{cases}$$

So we'll have $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \frac{4}{3} T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + \frac{5}{3} T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$, so then:

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \frac{4}{3}(2x) + \frac{5}{3}(1 - x) = \frac{8}{3}x + \frac{5}{3} - \frac{5}{3}x$$

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = x + \frac{5}{3}$$

The general background for this is that if you know the transformations of the vectors that form a basis for a domain, you can compute the transformation for any other vector in that domain (as it can be expressed as a linear combination of those basis vectors).

Problem 4: Let $T : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be defined by $T(A) = A + A^T$.

1. Is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \ker(T)$?

To check if the matrix is in the kernel, see if you get the 0 matrix from the transformation:

$$T(A) = T\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So A is in the kernel.

2. Find a basis for $\ker(T)$ and specify $\mathrm{nullity}(T)$

A is in $\ker(T)$ if and only if $A + A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}$$

Set that equal to zero to find the basis for the kernel

$$\rightsquigarrow \begin{cases} 2a = 0 \\ b + c = 0 \\ 2d = 0 \end{cases} \to \begin{cases} a = 0 \\ b = -c \\ d = 0 \end{cases}$$

So a basis for the kernel is

$$\mathrm{span}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

3. Find a basis for $\mathrm{range}(T)$ and specify $\mathrm{rank}(T)$

Lecture 22: more on diagonalization and complex eigenvalues

Recall that:

1. $A, B \in \text{Mat}(n \times n)(\mathbb{R})$ are similar if $B = P^{-1}AP$ for some P. Similarity creates a partition of $\text{Mat}_{n \times n}(\mathbb{R})$.

Example: Are $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ similar?

$$\det(A) = 1 - 4 = -3$$

$$\det(B) = 4 - 1 = 3$$

So A and B cannot be similar.

Note: remember that if matrices A and B have different determinants, they can't be similar, but if they have the same determinant, they're not necessarily similar (so it works one way but not the other)

2. A is diagonalizable if $A \sim D$ where D is diagonal.

Theorem: A is diagonalizable $\Leftrightarrow$ for each eiganvalue λ of A

If A is diagonalizable and has $\lambda_1, ..., \lambda_m$ eigenvalues (they may be repeated) with corresponding eigenvectors $v_1, ..., v_n$

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}^{-1}}_{P^{-1}} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}}_{P}$$

Theorem: $\{v_1, v_2, ..., v_n\}$ as shown above is a basis of $\mathbb{R}^n$ consisting of eigenvectors.

Corollary 1: A is diagonalizable $\Leftrightarrow$ A has n distinct eigenvectors.

Corollary 2: For $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\bar{x}) = A\bar{x}$, if A is diagonalizable, then there exists a basis B of $\mathbb{R}^n$ such that $[T]_{B \leftarrow B} := [T]_B$ is a diagonal matrix. Moreover, $B =$ columns of P.

Note: this example is, according to the professor, always a hard exam question

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$. Find a basis B (if possible) such that $[T]_B$ is diagonal.

Solution: We must determine if A is diagonalizable.

$$\text{char}_A(x) = |A - xI_2| = \begin{vmatrix} 1 - x & 3 \\ 3 & -1 - x \end{vmatrix} = -(1 - x)(1 + x) = -3 = -x^2 - 2$$

$$-(x^2 + 2) \neq 0 \text{ for } x \in \mathbb{R}$$

Hence A is not diagonalizable, so there is no basis B such that $[T]_B$ is a diagonal matrix.

But now what happens to matrices A with complex eigenvalues?

Theorem: Assume A is a 2×2 matrix over $\mathbb{R}$ and it has a complex eigenvalue $\lambda = a + bi \in \mathbb{C}$ with corresponding eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2$. Then

1. $\bar{\lambda} = a - bi$ is the second eigenvalue of A with corresponding eigenvector $\bar{x} = \begin{bmatrix} \bar{x_1} \\ \bar{x_2} \end{bmatrix}$
2. A is similar to a *scalar-rotation matrix.*

$$A = P^{-1} \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\substack{\text{scaling matrix by} \\ r = |\lambda| = \sqrt{a^2+b^2}}} \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\substack{\text{rotation by } \theta \\ \text{counterclockwise} \\ \theta = \arg(\lambda)}} P$$

Moreover,

$$P = \begin{bmatrix} \mathrm{Re} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \mathrm{Im} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix}$$

and after simplification

$$\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Lecture 23: orthogonality and orthogonal complements

The dot product

Definition: for $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$, the *dot product* or *innerproduct* is defined as

$$u \cdot v = u^T \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Example: Given $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, then $u \cdot v = (1)(-1) + (-1)(2) + (2)(3) = 3$

Properties: For all $u, v, w, \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

1. $u \cdot v = v \cdot u$ (commutativity)
2. $cU \cdot v = u \cdot cV = c(u \cdot v)$
3. $(u + v) \cdot w = w \cdot u + w \cdot v$ (distributivity)
4. $u \cdot u \geq 0$ (non-negativity)
5. $u \cdot u = 0 \Leftrightarrow u = \vec{0}$

Lengths/norms

Definition: The length or norm of $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$ is defined as:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + \ldots + u_n^2}$$

Example: Given $u = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $\|u\| = \sqrt{a^2 + b^2}$. In a way, we're using the dot product to generalize the Pythagorean theorem.

Definiton: A vector $u \in \mathbb{R}^n$ such that $\|u\| = 1$ is called a unit vector.

In $\mathbb{R}^2$, all unit vectors are on the circle with radius 1, in $\mathbb{R}^3$, all unit vectors are on the sphere with radius 1, etc.

Any vector can be converted to a unit vector by just stretching or shrinking it.

Theorem: For any $u \in \mathbb{R}^n, u \neq 0$, the vector $\frac{1}{\|u\|} \cdot u$ is a unit vector in the same direction of u.

Example: Find a unit vector in the same direction of $u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

Solution: $\|u\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$, so a unit vector in the same direction as u is

$$\frac{1}{\|u\|} u = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Distance

Definition: The distance between $u, v, \in \mathbb{R}^n$ is defined as

$$\text{dist}(u, v) = \|u - v\|$$

Example: Given $u = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$, then

$$\text{dist}(u, v) = \|u - v\| = \left\| \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\| = \sqrt{17}$$

Properties:

1. $\text{dist}(u, v) = \text{dist}(v, u)$
2. $\text{dist}(u, v) \geq 0$ and $\text{dist}(u, v) = 0 \Leftrightarrow u = v$
3. $\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$ (triangle inequality)

Orthogonality

Definition: The vectors $u, v \in \mathbb{R}^n$ are orthogonal or perpendicular if $u \cdot v = 0$ (so equivalently, if $\|u\|\|v\| \cos\theta = 0$)

The notation we use is $u \perp v$

Note: $u \perp v$ for all vectors $v \in \mathbb{R}^n$ if and only if $u = \vec{0}$.

Example: $u = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow u \cdot v = 3(-1) + 1(2) + 1(1) = 0$, so $u \perp v$.

Orthogonal sets

Definiton: a set of vectors $\{v_1, ..., v_k\} \in \mathbb{R}^n$ is an orthogonal set if $v_i \perp v_j$ for all $i \neq j$.

Example:

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$.

$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$.

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$. (But it's not a basis, because the zero vector is in there).

It turns out that every time you have a set of three nonzero orthogonal vectors in $\mathbb{R}^3$, they're linearly independent and form a basis.

Orthogonal bases

Theorem: Given $B = \{u_1, ..., u_k\}$, an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then B is linearly independent, so B is a basis for $W = \operatorname{span}(u_1, ..., u_k)$. The basis B is called an *orthogonal basis* for W. Moreover, if in addition, $\|u_i\| = 1$, then B is called an *orthonormal basis* for W.

For example, $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$. We can convert any orthogonal basis into an orthonormal basis by doing the rescaling the vectors:

Example: $\left\{ u = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, w = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} \right\}$ is an orthogonal basis of $\mathbb{R}^3$, but it's not an orthonormal basis, since $\|u\| = \sqrt{9+1+1} = \sqrt{11} \neq 1$, so at least one of its vectors doesn't have a length of 1.

However,

$$\left\{ \frac{1}{\|u\|} u, \frac{1}{\|v\|} v, \frac{1}{\|w\|} w \right\} = \left\{ \frac{1}{\sqrt{71}} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{\frac{66}{4}}} \begin{bmatrix} -\frac{1}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} \right\}$$

is an orthonormal basis.

Lecture 24: orthogonal complements and projections

Orthogonal complements

Definiton: Let $W \leq \mathbb{R}^n$ be a subspace. The orthogonal complement of W is

$$W^\perp = \{x \in \mathbb{R}^n : w \cdot x = 0 \text{ for all } w \in W\}$$

Example: Given a plane W in $\mathbb{R}^3$ through the origin, $W^\perp$ is a line in $\mathbb{R}^3$ through the origin.

Algebraically, you can assume $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y - 3z = 0 \right\}$. Then you can do

$$0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = x + y - 3z$$

This shows that $W^\perp = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right)$, a line in $\mathbb{R}^3$.

Theorem: $W^\perp$ is a subspace of $\mathbb{R}^n$, and $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n) = n$.

Theorem: For any $m \times n$ matrix A:
1. $(\text{null}(A))^\perp = \text{row}(A) \subseteq \mathbb{R}^n$
2. $(\text{col}(A))^\perp = \text{null}(A^T) \subseteq \mathbb{R}^m$

Insert here: the four fundamental subspaces

Example: Let $W = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \subseteq \mathbb{R}^3$. Find a basis of $W^\perp$.

Solution: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. Observe that $W = \text{col}(A)$. So in order to compute a basis for $W^\perp$, we need to compute a basis for $(\text{col}(A))^\perp = \text{null}(A^T) \subseteq \mathbb{R}^3$.

Now,

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{cases} x - y = 0 \\ 2y + z = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ y = -\frac{1}{2}z \end{cases}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}z \\ -\frac{1}{2}z \\ z \end{bmatrix} = z \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow \text{null}(A^T) = \text{span}\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)$$

Hence $W^\perp = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)$, and a basis is $B = \left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

Orthogonal projections

Theorem: The orthogonal projection of $V \in \mathbb{R}^n$ onto $u \in \mathbb{R}^n, u \neq 0$, is

$$\text{proj}_u(v) = \frac{u \cdot v}{\|u\|^2} * u$$

Moreover,

$$V = \underbrace{\text{proj}_u(v)}_{\substack{\text{vector parallel} \\ \text{to u which is in} \\ \text{L = span(u)}}} + \underbrace{\text{perp}_u(v)}_{\substack{\text{vector orthogonal} \\ \text{to u which is in} \\ \text{L\^{}perp}}}$$

A formula for $\text{perp}_u(v)$ is

$$\text{perp}_u(v) = v - \text{proj}_u(v)$$

The distance from V to L is $\|\text{perp}_u(v)\|$

Example: Let $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Find the distance from V to $L = \text{span}(u)$.

Solution: First we compute $\text{proj}_u(v)$:

$$\text{proj}_u(v) = \frac{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{3}{(1)^2 + (-1)^2 + (2)^2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Next, we compute $\text{perp}_u(v)$:

$$\text{perp}_u(v) = V - \text{proj}_u(v) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Last, we compute $\|\text{perp}_u(v)\|$:

$$\|\text{perp}_u(v)\| = \left\| \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \right\| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 0^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

This is the distance from V to L.

The orthogonal decomposition theorem

! He said "mark this topic as 'I need to master it for the final'" so...

Theorem: Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $B = \{u_1, ..., u_k\}$. For every $V \in \mathbb{R}^n$, the orthogonal projection of V onto W is

$$\mathrm{proj}_W(V) = \mathrm{proj}_{u_1}(V) + ... + \mathrm{proj}_{u_k}(V)$$

The orthogonal complement of V to W is defined as

$$\mathrm{perp}_W(V) = V - \mathrm{proj}_W(V)$$

The distance from V to W is

$$\mathrm{dist}(V, W) = \|\mathrm{perp}_W(V)\|$$

Also, the vector V can be written as

$$V = \mathrm{proj}_W(V) + \mathrm{perp}_W(V)$$

and this is called an orthogonal decomposition.

(This is pretty visual, so it's probably good to look at some diagrams as well)

Example: Let $W = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right) \subseteq \mathbb{R}^3$ and $V = \begin{bmatrix}3\\-1\\2\end{bmatrix} \in \mathbb{R}^3$. Find $\text{proj}_W(V)$ and $\text{perp}_W(V)$

Solution: W is a subspace of $\mathbb{R}^n$ and its basis $B = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ is an orthogonal basis (since the dot product of its two vectors is 0), so we can apply the orthogonal decomposition theorem. *(Don't forget to double-check these creiteria before using it!)*

Hence,

$$\text{proj}_W(V) = \text{proj}_{u_1}(V) = \text{proj}_{u_2}(V)$$

$$= \frac{u_1 \cdot V}{\|u_1\|^2}u_1 + \frac{u_2 \cdot V}{\|u_2\|^2}u_2$$

$$= \frac{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}3\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\0\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\begin{bmatrix}-1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}3\\-1\\2\end{bmatrix}}{\left\|\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\|^2}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$= \frac{2}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + -\frac{2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{5}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}$$

Now we can compute

$$\text{perp}_W(V) = V - \text{proj}_W(V)$$

$$= \begin{bmatrix}3\\-1\\2\end{bmatrix} - \frac{1}{3}\begin{bmatrix}5\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{4}{3}\\-\frac{4}{3}\\\frac{8}{3}\end{bmatrix}$$

And there's our answer!

Lecture 25: Gram-Schmidt algorithm and QR factorization

Gram-Schmidt algorithm

- Input: $B = \{u_1, ..., u_k\}$ basis for $W \subseteq \mathbb{R}^n$
- Output: orthogonal basis for W with $B = \{v_1, ..., v_k\}$
- Steps:
  1. Set $v_1 = u_1$
  2. Recursively apply the following fomula for k vectors:

$$v_k = u_k - \frac{v_1 \cdot u_k}{\|v_1\|^2} v_1 - ... - \frac{v_{k-1} \cdot u_k}{\|v_{k-1}\|^2} v_{k-1}$$

  For example, the first couple iterations of this second step:

  1. $v_2 = u_2 - \frac{v_1 \cdot u_2}{\|v_1\|^2} v_1$
     - so $v_2 = \text{perp}_{W_1}(v), W_1 = \text{span}(u_1) = \text{span}(v_1)$
  2. $v_3 = u_3 - \frac{v_1 \cdot u_3}{\|v_1\|^2} v_1 - \frac{v_2 \cdot u_2}{\|v_2\|^2} v_2$
  3. Repeat this

Example: Let $W = \operatorname{span}(u_1, u_2, u_3) \subseteq \mathbb{R}^4$, where $u_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$. Apply the Gram-Schmidt algorithm to compute an orthogonal basis for W.

Solution:

1) Set $v_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

2) (using a tilde because we'll modify v_2)

$$\tilde{v}_2 = u_2 - \frac{v_1 \cdot u_k}{\|v_1\|^2} v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

The fraction here would make the subsequent calculations annoying — but since we're just looking for a basis (not a vector space), it's fine to scale the vector, since you'll still get the same span. So we can just ignore the fraction for this!

Note: in formal terms, scaling doesn't modify orthogonality because $u \cdot v = 0 \Leftrightarrow u \cdot (cv) = 0$, where c is a constant. Moreover, $\operatorname{span}(u, v) = \operatorname{span}(u, cv)$.

Hence, by scaling, we can take $v_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.

3)

$$\tilde{v}_3 = u_3 - \frac{v_1 \cdot u_3}{\|v_1\|^2} v_1 - \frac{v_2 \cdot u_3}{\|v_2\|^2} v_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Scaling like before, we have $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

After all that, we have

$$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

and an orthogonal basis for W is $\operatorname{span}(u_1, u_2, u_3)$.

Note: if we'd like to compute an orthonormal basis, first apply GS, then normalize the vectors.

Lecture 26: QR factorization

QR factorization

Definition: Let A be an $m \times n$ matrix with linearly independent columns (so $m \geq n$). Then applying Gram-Schmidt to $W = \text{col}(A)$ yields

$$A = \underbrace{Q}_{\substack{m\times n \\ \text{col}(A) = \text{col}(Q) \\ \text{it has orthonormal} \\ \text{columns}}} \underbrace{R}_{\substack{n\times n \\ \text{it is invertible} \\ \text{and always triangular}}}$$

Applications:

1. Numerical approximation of eigenvalues
2. The problem of least squares approximation

Note: Q has orthonormal columns, so

$$Q^T Q = I_n$$

Hence, if we know A and Q, then R is computed as $R = Q^T A$, so the work consists of computing Q.

Example: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find the QR factorization of A.

Solution: We verify that A has linearly independent columns:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\operatorname{rank}(A) = 3 =$ # columns of A $\Rightarrow$ all columns of A are linearly independent.

We apply Gram-Schmidt to $W = \operatorname{col}(A) = \operatorname{span}(u_1, u_2, u_3)$, where $u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Computing this:

1) $v_1 = u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

2) $v_2 = u_2 - \frac{v_1 \cdot u_2}{\|v_1\|^2} v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

3) $v_3 = u_3 - \frac{v_1 \cdot u_3}{\|v_1\|^2} v_1 - \frac{v_2 \cdot u_3}{\|v_2\|^2} v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\left(\frac{1}{2}\right)^2 \left\| \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\|^2} \left(-\frac{1}{2}\right) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} -$
$\frac{1}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Hence, an orthogonal basis for $\operatorname{col}(A)$ is $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, v_3 = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. We now normalize these vectors so as to get an orthonormal basis.

So to compute that:

$$\tilde{v}_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\tilde{v}_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\frac{1}{2} \left\| \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\|} \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\tilde{v}_3 = \frac{1}{\|v_3\|} v_3 = \frac{1}{\frac{2}{3} \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\|} \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Hence:

$$Q = \begin{bmatrix} | & | & | \\ \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

We also would like to compute the matrix R. To do so, we can use $R = Q^T A$. So then

$$R = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} * \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

This is an upper triangular matrix! (It always will be.)

Lecture 27: least squares approximation

Least squares approximation

Motivating example: Find a curve that best fits a set of data points.

Let's use data points $(1, 2), (2, 2), (3, 4)$. Assume we would like a line $y = ax + b$ that best fitst these points. The line should minimize

$$\|\varepsilon\|^2 = \left\| \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \right\|^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2$$

where ε is the vertical distance from the point to the line. Plugging the points into $y = ax + b$:

$$\begin{cases} a + b = 2 \\ 2a + b = 2 \\ 3a + b = 4 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

This gives us an equation in the form $Ax = b$.

Definition: If A is an $m \times n$ matrix and $b \in \mathbb{R}^n$, a least square solution to $Ax = b$ is a vector $\tilde{x} \in \mathbb{R}^n$ such that

$$\|b - A\tilde{x}\| \leq \|b - Ax\|, \forall x \in \mathbb{R}^n$$

Note: $\forall$ means "for all"

Theorem: First, $Ax = b$ has at least one least square solution.

Moreover,

1) $\tilde{x}$ is a least-squares solution to $Ax = b \Leftrightarrow \tilde{x}$ is a solution to the normal equation $\underbrace{A^T A}_{\text{square matrix}} x = \underbrace{A^T b}_{\text{column vector}}$

2) The solution is unique $\Leftrightarrow A^T A$ is invertible $\Leftrightarrow A$ has linearly independent columns

For example, assume A is 6×5 and $\operatorname{rank}(A) = 5$. Does there exist a unique least-squares solution to $Ax = b$ for any b? $\rightsquigarrow$ yes

3) When there is a unique solution, it can be found:

$$\tilde{x} = \left(A^T A\right)^{-1} A^T b$$

Example: Returning to the above motivating example: for the points $(1, 2), (2, 2), (3, 4)$, find a line $y = ax + b$ that best approximates the data points.

$$\begin{cases} a + b = 2 \\ 2a + b = 2 \\ 3a + b = 4 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

This is in the form $Ax = b$. Solving the normal equation $A^T Ax = A^T b$:

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}; A^T b = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \end{bmatrix}$$

We want to solve

$$\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \end{bmatrix} \xLeftrightarrow[]{\substack{\text{Gauss-Jordan} \\ \text{or compute } (A^T A)^{-1}}} \tilde{x} = \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix} \begin{bmatrix} 18 \\ 8 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then the least squares solution is $\tilde{x} = \begin{bmatrix} \frac{3}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}$.

Now let's compute the error of this approximation:

$$\|b - A\tilde{x}\| = \left\| \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 5 \\ 8 \\ 11 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right\| = \frac{1}{3} \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\| = \frac{1}{3}\sqrt{6}$$

Ok but what if we added another point and a parabola would better fit? Now we would like to use $y = a + bx + cx^2$. Plugging in the data points:

$(-1,1)$	$a - b + c = 1$
$(0,-1)$	$a = -1$
$(1,0)$	$a + b + c = 0$
$(2,2)$	$a + 2b + 4c = 2$

and that becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

And then solving the normal equation $A^T Ax = A^T b$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$$

Using Gaussian elimination we solve:

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 6 & 2 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 9 \end{array}\right] \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{7}{10} \\ -\frac{3}{5} \\ 1 \end{bmatrix}$$

Then the least square solution is

$$\tilde{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{7}{10} \\ -\frac{3}{5} \\ 1 \end{bmatrix}$$

which is equivalent to saying that the parabola is

$$y = -\frac{7}{10} - \frac{3}{5}x + x^2$$

Least squares and QR factorization

Theorem: Let A be a matrix $m \times n$ with linearly independent columns and $b \in \mathbb{R}^n$. If $A = QR$ is the QR-factorization of A, then the LS solution to $Ax = b$ is:

$$\tilde{x} = R^{-1} Q^T b$$

or equivalently

(nevermind lol)

Lecture 28: intro to differential equations

For the differential equations part of this class, we'll be covering chapters 1-5 of "A First Course in Differential Equations with Applications", 11th edition, by D. Zill.

What is a differential equation? A differential equation is an equation involving one or more functions and their derivatives, with one or more independent variables.

Classification of differential equations can be done according to type, order, and linearity.

Classification by type

1. Ordinary differential equations (ODE)
   - There is only one independent variable
   - e.g. $y' + y = 5e^x$
     - so $y' = \frac{dy}{dx}$ where y is the dependent variable and x is the independent variable
   - e.g. $\frac{dx}{dt} + \frac{dy}{dt} = 3x + y$
     - so t is independent and x, y are dependent
2. Partial differential equations (PDE)
   - There are at least two independent variables
   - e.g. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
     - x, y are independent and u is a function of x, y

Classification by order

Definiton: The order of a differential equation is the highest derivative in the equation.

Examples:

$y'' + 5y = 3e^x + \sin(x)$ is a second-order ordinary differential equation because we have the second derivative of y.

$(y''')^2 + (y'')^4 + (y')^6 = e^x$ is a third-order ODE since there's the third derivative in there.

$y'y + y = \sin(x)$ is a first-order ODE.

In general, an n^{th}-order ODE can be written in the form

$$F(x, y, y', ..., y^{(n)} = 0$$

For example, $y'' + 5y = 3e^x - \sin(x) = 0$.

Often (and always for this course), we can solve $F\left(x, y, y', ..., y^{(n)}\right)$ for $y^{(n)}$ with

$$y^{(n)} = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = f\left(x, y, y', ..., y^{(n-1)}\right)$$

This is the normal form of the n^{th}-order ODE.

Example: $y'' + 5y = 3e^x + \sin(x)$ has normal form

$$y'' = \underbrace{3e^x + \sin(x) - 5y + 0y'}_{f(x,y,y')}$$

(obviously the $0y'$ is optional, it's just there for clarity)

Example: What is the normal form of $y - x + 4xy' = 0$?

Solution: $y' = \frac{x-y}{4}x$

Another important form for a first-order ODE is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$

Example: What is the differential form of $y - x + 4xy' = 0$?

Solution: We replace y' by $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$y - x + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

then multiply by $\mathrm{d}x$:

$$(y - x)\,\mathrm{d}x + 4x\,\mathrm{d}y = 0$$

(Here, $M(x, y) = y - x$ and $N(x, y) = 4x$)

Classification by linearity

- Differential equations
  - Nonlinear
  - Linear
    - Homogenous: $g(x) = 0$
    - Non-homogenous: $g(x) \neq 0$

Definition: An n^{th}-order ODE is said to be liniear if it is of the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Note:

1. $y, y', \ldots, y^{(n)}$ have power 1.
2. The coefficients of $y, y', \ldots, y^{(n)}$ are functions of x only
3. $g(x)$ depends on x only

Example: Is $(y - x)\,\mathrm{d}x + 4x\,\mathrm{d}y = 0$ linear or not?

Solution: Dividing by $\mathrm{d}x$,

$$y - x + 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \underbrace{4x}_{a_1(x)}y' + \underbrace{}_{a_0(x)}y = \underbrace{x}_{g(x)}$$

(That blank $a_0(x)$ isn't a typo; there's just nothing there)

So yes, it is linear!

Example: Is $(1 - y)y' + 2y = e^x$ linear? ⇝ No, because the coefficient of y' is not a function of x only.

Example: Is $\frac{\mathrm{d}^2 y}{\mathrm{d}x}$

Lecture 29: solutions to ODE, IVP, direction fields, and autonomous

Solutions to ODEs

Definiton: Given an ODE $F\left(x, y, y', ..., y^{(n)}\right) = 0$, a solution is any function $\varphi : I \to \mathbb{R}$ (I is an interval $(a, b), (-\infty, a), (a, \infty), (-\infty, +\infty)$) and posessing at least n continuous derivatives such that

$$F\left(x, \varphi(x), \varphi'(x), ..., \varphi^{(n)}(x)\right) = 0$$

Note: a solution comes along an interval.

Solutions to ODEs can be given explicitly or implicitly:

Explicit solutions

Explicit solutions are in the form $y = f(x)$, where the dependent variable is given in terms of the independent variable only. In other words, y is a function of x only.

Example: for $y'' - 2y' + y = 0$, which is a second-order homogenous linear ODE, an explicit solution is $y = e^x + xe^x, x \in (-\infty, \infty)$.

We can also check that this is indeed a solution:

$$y' = e^x + e^x + xe^x = 2e^x + xe^x$$
$$y'' = 2e^x + e^x + xe^x = 3e^x + xe^x$$

So

$$(3e^x + xe^x) - 2(2e^x + xe^x) + (e^x + xe^x) = 0$$

Yep, this is true! So it's a solution.

Implicit solutions

Implicit solutions are in the form $G(x, y) = 0$, which is a solution curve. y is *not* a function of x.

Example: For $y'' = 2y(y')^3$, an implicit solution is $y^3 + 3y = 1 - 3x, x \in (-\infty, \infty)$.

Checking this requires that we apply implicit differentiation:

For the first derivative, we have

$$3y^2y' + 3y' = -3$$

and then solving for y':

$$y' = \frac{-3}{3y^2 + 3}$$

For the second derivative, we have

$$6yy'y' + 3y^2y'' + 3y'' = 0$$

and then solving for y'':

$$y'' = \frac{-6y(y')^2}{3y^2 + 3}$$

Now we must check that $y'' = 2y(y')^3$. If you solve this out, it holds, so this is a solution.

Initial value problems

An ODE has infinitely many solution curves (in general). For example, for $\frac{dy}{dx} = \frac{-x}{y}$ we have the solution $x^2 + y^2 = C$, which equates to a circle of any radius. So how do we choose a particular solution? We consider initial value problems:

Solve

$$\frac{d^n y}{dx^n} = f\left(x, y, ..., y^{(n-1)}\right)$$
$$\text{subject to } y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1}$$

Example: For $y'' - 2y' + y = 0$, the general solution is

$$y = c_1 e^x + c_2 e^x \text{ on } (-\infty, \infty), c_1, c_2, \in \mathbb{R}$$

and then we have

$$y' = c_1 e^x + c_2 e^x + c_2 x e^x = (c_1 + c_2) e^x + c_2 x e^x$$

c_1 and c_2 are parameters, so we have a 2-parameter set of solutions.

Now, consider the IVP in which we have $y'' - 2y' + y = 0$ subject to $y(0) = 1, y'(0) = 2$. Since we know the general solution, all we have to do now is plug in the initial conditions.

Plugging in:

$$y(0) = 1 \Rightarrow (x, y) = (0, 1) \Rightarrow 1 = c_1 e^0 + c_2 0 e^0 \Rightarrow 1 = c_1 + 0 c_2$$

$$y'(0) = 2 \Rightarrow (x, y) = (0, 2) \Rightarrow 2 = (c_1 + c_2) e^0 + c_2 0 e^0 \Rightarrow 2 = c_1 + c_2$$

And now we have a system of linear equations! Solving that with $c_1 = 1, c_2 = 1$, the particular solution is

$$y = e^x + x e^x \text{ on } (-\infty, \infty)$$

Direction fields

For first-order ODEs in normal form $\frac{dy}{dx} = f(x, y)$, a direction field is a graphical representation of the slopes of the solution curves.

A particular case is $\frac{dy}{dx} = f(y)$, which is an autonomous equation.

A critical point of $\frac{dy}{dx} = f(y)$ is any real number C such that $f(c) = 0$.

Example: For $\frac{dy}{dx} = \sin(y)$, the critical points are where $\sin(y) = 0$, so they are any point with $y = k\pi, k \in \mathbb{Z}$.

We use the critical points to graph a direction field of an autonomous equation.

Lecture 30: ??? I missed this one to watch the eclipse

Lecture 32: exact equations

Definiton: a first-order ODE $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ (which is differential form) is called exact if ∃ (there exists) a differential function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M(x,y)$, $\frac{\partial f}{\partial y} = N(x,y)$. In this case, $\underbrace{\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y}_{\text{Increment of } f \text{ as we move from } (x,y) \text{ to } (x+\mathrm{d}x, y+\mathrm{d}y)} = 0$. An implicit solution is $f(x,y) = c$.

Method of solution (how to find $f(x,y)$):

1. Integrate $M(x,y)$ with respect to x.

$$f(x,y) = \int M(x,y)\,\mathrm{d}x + g(y)$$

where $g(y)$ is an arbitrary function of y.

2. Differentiate with rexpect to y and set equal to $N(x,y)$, then solve for $g'(y)$.

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)\,\mathrm{d}x\right)$$

That's the formula, but it's easier to just remember the workflow instead of trying to memorize it.

3. Integrate $g'(y)$ with respect to y.

$$g(y) = \int g'(y)\,\mathrm{d}y$$

Then substitute the equation in step 1.

Remember that a particular solution is $f(x,y) = c$ — with a constant, not just this function! We can compute the constant from initial values.

Example: Is $2xy\,\mathrm{d}x + (x^2 - 1)\,\mathrm{d}y = 0$ an exact equation? If so, solve.

Solution: $M(x,y) = 2xy, N(x,y) = x^2 - 1$.

Then we apply the criterion of exactness:

$$\frac{\partial M}{\partial y} = 2x \quad \frac{\partial N}{\partial x} = 2x$$

They're equal, so the equation is exact.

Now to solve!

$$f(x,y) = \int 2xy\,\mathrm{d}x + g(y)$$

$$\Rightarrow f(x,y) = x^2 y + g(y)$$

then do

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

We set this equal to $N(x,y) = x^2 - 1$

$$x^2 - 1 = x^2 + g'(y)$$

$$\Rightarrow g'(y) = -1$$

Now we integrate this:

$$g(y) = \int d'(y)\,\mathrm{d}y = \int -1\,\mathrm{d}y = -y$$

and plugging that back into the equation from before

$$f(x,y) = x^2 y - y$$

Now we have our function, but we need to remember to set it equal to a constant! Hence, the final answer is

$$x^2 y - y = C \text{ for some } C \in \mathbb{R}$$

Solving this for y, we have

$$y = \frac{C}{x^2 - 1}$$

so any of the intervals $(-\infty, -1), (-1, 1), (1, \infty)$ can be chosen as an interval of solution.

All these intervals are correct — unless we have an initial value problem.

Let's see this with the case where $y(3) = 2$, so $(x = 3, y = 2)$.

Plugging this into the general solution,

$$2 = \frac{C}{9-1} = \frac{C}{8} \Rightarrow C = 16$$

The solution is

$$y = \frac{16}{x^2 - 1}, x \in (1, \infty)$$

We choose the interval based on the x value.

Example: solve the following:

$$\left(e^{2y} - y\cos(xy)\right)\mathrm{d}x + \left(2xe^{2y} - x\cos(xy) + 2y\right)\mathrm{d}y = 0$$

Solution: Firstly, is this equation exact?

$$M(x,y) = e^{2y} - y\cos(xy)$$
$$N(x,y) = 2xe^{2y} - x\cos(xy) + 2y$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - (\cos(xy) + y(-\sin(xy)x))$$

$$\frac{\partial N}{\partial x} = 2e^{2y} - (\cos(xy) + x(-\sin(xy)y))$$

These two are equal, so the equation is exact!

Now solving:

$$1)\ f(x,y) = \int \left(e^{2y} - y\cos(xy)\right)\mathrm{d}x + g(y)$$

$$\Rightarrow f(x,y) = xe^{2y} - \sin(xy) + g(y)$$

$$2)\ \frac{\partial f}{\partial y} = 2xe^{2y} - x\cos(xy) + g'(y) = N(x,y)$$

Solving for $g'(y)$:

$$g'(y) = N(x,y) - 2xe^{2y} + x\cos(xy) = 2y$$

$$3)\ g(y) = \int g'(y)\,\mathrm{d}y = \int 2y\,\mathrm{d}y = y^2$$

Hence:

$$f(x,y) = xe^{2y} - \sin(xy) + y^2$$

So the soliution to the differential equation is

$$xe^{2y} - \sin(xy) + y^2 = C \text{ for some } c \in \mathbb{R}$$

Note: this solution represents a family of implicit solutions.

Lecture 32: more on exact equations and linear models

From last time...

Definition: $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is exact if ∃ a function f such that $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$.

Criterion for exactness: $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Integrating factors

Sometimes a non-exact equation can be made exact by finding an integrating factor. Here's how:

Assume $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is non-exact. Then, look at

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

1) If this is a function of x only, then the integrating factor is

$$\mu(x) = \exp\left(\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) \mathrm{d}x\right)$$

2) If this is a function of y only, then the integrating factor is

$$\mu(y) = \exp\left(\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right) \mathrm{d}x\right)$$

In either case, we can then turn $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ into an exact equation by multiplying it by the integrating factor.

Example: Solve $(2y^2 + 3x)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0$.

Solution: First, check for exactness:

$$\begin{cases} M(x,y) = 2y^2 + 3x \\ N(x,y) = 2xy \end{cases} \rightsquigarrow \begin{cases} \frac{\partial M}{\partial y} = 4y \\ \frac{\partial N}{\partial x} = 2y \end{cases}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \therefore \begin{matrix}\text{the equation} \\ \text{is not exact}\end{matrix}$$

Now, let's check if we can make the equation exact:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}$$

✓ we can make the equation into an exact equation!

Ok so finding the integrating factor:

$$\mu(x) = \exp\left(\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) \mathrm{d}x\right) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln|x|}$$

Because of the absolute value here, we have to do some further analysis.

If we select $x \in (0, \infty)$, then $|x| = x$, so $\mu(x) = e^{\ln|x|} = e^{\ln x} = x \ \forall x \in (0, \infty)$
(remember "$\forall$" means "for all", so $(0, \infty)$ is the interval of the solution)

We now multiply the ODE by the integrating factor $\mu(x) = x$:

$$x((2y^2 + 3x)\,\mathrm{d}x + (2xy)\,\mathrm{d}y = 0)$$

$$(2xy^2 + 3x^2)\,\mathrm{d}x + 2x^2y\,\mathrm{d}y = 0$$

$$\rightsquigarrow \begin{cases} M(x,y) = 2xy^2 + 3x^2 \\ N(x,y) = 2x^2y \end{cases}$$

$$\rightsquigarrow \begin{cases} \frac{\partial M}{\partial y} = 4xy \\ \frac{\partial N}{\partial x} = 4xy \end{cases} \therefore \begin{matrix}\text{the equation} \\ \text{is exact}\end{matrix}$$

Solving the new equation:

1)

$$f(x,y) = \int M(x,y)\,\mathrm{d}x + g(y)$$

$$f(x,y) = \int (2xy^2 + 3x^2)\,\mathrm{d}x + g(y)$$

$$f(x,y) = x^2y^2 + x^3 + g(y)$$

2) differentiate and set equal to $N(x,y)$

$$\frac{\partial f}{\partial y} = 2x^2y + g'(y) = 2x^2y$$

3) then

$$g'(y) = 0 \Rightarrow g(y) = C \text{ for some } C \in \mathbb{R}$$

Hence

$$f(x, y) = x^2 y^2 + x^3 + C$$

So the family of solutions is

$$x^2 y^2 + x^3 + C = K \text{ for } C, K \in \mathbb{R}$$

$$\Rightarrow x^2 y^2 + x^3 = \bar{C} \text{ for } \bar{C} \in \mathbb{R}$$

and the interval of the solution is $(0, \infty)$.

Linear models (§ 3.1)

Some real-life applications have a first-order ODE as a mathematical model:

1. Growth and decay: $\frac{\mathrm{d}P}{\mathrm{d}t} = kP, P(0) = P_0$
   - Growth if $k > 0$, decay if $k < 0$
   - We can rewrite this as a first-order linear ODE, $P' - kP = 0$, which we can solve with an integrating factor
2. Newton's Law of Cooling/Warming: $\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_m)$
   - $T(t)$ is the temperature in time of a given object at time t
   - T_m is the temperature of the environment (surrounding the object), which we assume is constant
3. Mixtures of solutions, like brine (salt & water): $\frac{\mathrm{d}A}{\mathrm{d}t} = R_{\text{in}} - R_{\text{out}}$
   - $A(t)$ is the amount of salt (by mass) in the tank at time t
   - The rate at which $A(t)$ changes is a net rate
     - R_{in} is the input rate of salt; R_{out} is the output rate
     - They're usually computed like this:
       - $R_{\text{in}} =$ (concdntration of salt in inflow)$*$(input rate of brine)
       - $R_{\text{out}} =$ (concentration of salt in outflow)$*$(output rate of brine)

Lecture 33: mixtures, second-order ODEs, Wronskians

Mixture example

Example: A tank initially contains 50 L of water and 20 g of salt. Water containing a salt concentration of 2 g/L enters the tank at a rate of 5 L/min, and the well-stirred solution leaves the tank at the same rate.

a) Find an expression for the amount of salt in the tank at time t.

Solution:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = R_{\text{in}} - R_{\text{out}} = \left(2\frac{g}{L}\right)\left(5\frac{L}{\text{min}}\right) - \left(\frac{A(t)}{50L}\right)\left(\frac{5L}{\text{min}}\right) = 10\frac{g}{\text{min}} - \frac{A(t)}{10}\frac{g}{\text{min}}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 10 - \frac{A(t)}{10} \Leftrightarrow \frac{\mathrm{d}A}{\mathrm{d}t} + \frac{A(t)}{10} = 10 \Leftrightarrow \frac{\mathrm{d}A}{\mathrm{d}t} + \underbrace{\frac{1}{10}}_{P(t)} A(t) = 10$$

Integrating factor:

$$\mu(t) = e^{\int P(t)\,\mathrm{d}t} = e^{\int \frac{1}{10}\,\mathrm{d}t} = e^{\frac{1}{10}t}$$

Then multiply by $\mu(t)$:

$$e^{\frac{1}{10}t}\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{1}{10}e^{\frac{1}{10}t}A(t) = e^{\frac{1}{10}t} * 10$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{1}{10}t}A(t)\right) = 10e^{\frac{1}{10}t}$$

Integrating both sides:

$$e^{\frac{1}{10}t}A(t) = \int 10e^{\frac{1}{10}t}\,\mathrm{d}t$$

$$e^{\frac{1}{10}t}A(t) = 100e^{\frac{1}{10}t} + C$$

$$\Rightarrow A(t) = \frac{100e^{\frac{1}{10}t}}{e^{\frac{1}{10}t}} + \frac{C}{e^{\frac{1}{10}t}})$$

hence

$$A(t) = 100 + Ce^{-\frac{1}{10}t}$$

$$\forall t \in (0, \infty)$$

This is a general solution. To find C, we need to solve an initial value problem. From the problem statement,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 10 - \frac{1}{10}A(t) \text{ subject to } A(0) = 20\frac{g}{L}$$

$$20 = A(0) = 100 + Ce^{-\frac{1}{10}(0)}$$

$$20 = 100 + C$$

$$C = -80$$

$$A(t) = 100 - 80e^{-\frac{1}{10}t} \forall t \in (0, \infty)$$

b) How long does it take for the amount of salt to reach 60 grams?

Solution: We want time $t > 0$ such that $60 = A(t) = 199 - 80e^{-\frac{1}{10}t}$

$$\Rightarrow 60 = 100 - 80e^{-\frac{1}{10}t} \Rightarrow \frac{60 - 100}{-80} = e^{-\frac{1}{10}t} \Rightarrow \ln\left(\frac{-40}{-80}\right) = \ln\left(e^{-\frac{1}{10}t}\right) \Rightarrow \ln\left(-\frac{1}{2}\right) = -\frac{1}{10}t$$

$$\Rightarrow t = -10\ln\left(\frac{1}{2}\right)$$

c) Find the approximate amount of salt after 100 years.

This can be modeled by taking the limit as $t \to \infty$:

$$\lim_{t\to\infty} A(t) = 100g$$

§ 4.1 homogenous equations in the textbook

Recall that

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \dots + a_0(x)y = g(x)$$

is an n^{th}-order ODE that is linear.

The associated homogenous equation is the same thing, except $g(x) = 0$.

We will see that a general solution to the above equation is

$$\begin{matrix}\text{general solution of} \\ \text{homogenous equation}\end{matrix} + \begin{matrix}\text{particular solution of} \\ \text{nonhomogenous equation}\end{matrix}$$

Example: $2y'' + 3y' - 5y = e^x$, which is a non-homogenous second-order ODE.

The associated homogenous equation is $2y'' + 3y' - 5y = 0$ which is a homogenous linear second-order ODE.

Superposition principle

Theorem: If $y_1, \dots, y_k$ are solutions to a homogenous equation, then

$$y = C_1 y_1 + \dots + C_k y_k$$
$$C_i \in \mathbb{R}$$

is a solution. So

$$\operatorname{span}(y_1, \dots, y_k) = W$$

where W is a subspace of $F =$ all functions $\mathbb{R} \to \mathbb{R}$.

Example: $y_1 = x^2$, $y_2 = x^2 \ln(x)$, $x \in (0, \infty)$ are solutions to $x^3 y''' - 2xy' + 4y = 0$, which is a homogenous third-order linear ODE.

We can verify that $y_1 = x^2$ and $y_2 = x^2 \ln(x)$ are solutions of this ODE.

By the superposition principle,

$$y = c_1 x^2 + c_2 x^2 \ln(x), x \in (0, \infty), c_1, c_2 \in \mathbb{R}$$

is also a solution to this ODE.

(end of class)

Lecture 34: Wronskians and fundamental sets of solutions

Linear independence

Definition: $f_1, f_2, ..., f_n$ are linearly independent on an interval I if:

$$c_1 f_1(x) + ... + c_n f_n(x) = 0 \forall x \in I$$
$$\text{where } c_1 = c_2 = ... = c_n = 0$$

Example: Are $f_1(x) = \sin(2x), f_2(x) = \sin(x)\cos(x), I = (-\infty, \infty)$ linearly independent?

Solution: use the zero function

$$c_1 f_1(x) + c_2 f_2(x) = 0 \forall x \in (-\infty, \infty)$$

$$c_1 \sin(2x) + c_2 \sin(x)\cos(x) = 0 \forall x \in (-\infty, \infty)$$

Recall that $\sin(2x) = 2\sin(x)\cos(x) \forall x \in (-\infty, \infty)$ Hence,

$$1 \sin(2x) + -2 \sin(x)\cos(x) = 0$$

$c_1, c_2 \neq 0 \Rightarrow$ the functions are *not* linearly independent.

Example: Are $f_1(x) = |x|, f_2(x) = x, I = (-\infty, \infty)$ linearly independent?

Solution: We consider

$$c_1 \ |x| + c_2 x = 0 \forall x \in (-\infty, \infty)$$

If this is true for all $x \in (-\infty, \infty)$ in particular, the equation holds for $x = 1$ and $x = -1$.

$$\begin{cases} x = 1 : c_1 + c_2 = 0 \\ x = -1 : c_1 - c_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = -2$, the system has a trivial solution of $c_1 = c_2 = 0$, so the functions are linearly independent.

Note: The interval matters. Consider

$$f_1(x) = |x|, f_2(x) = x \forall x \in (1, \infty)$$

Are $f_1(x)$ and $f_2(x)$ linearly independent? Nope!

$$f_1(x) = |x| = x = f_2(x)$$

so

$$1 f_1(x) + (-1) f_2(x) = 0 \forall x \in (1, \infty)$$

so they're not linearly independent on this interval.

Wronskians

Suppose $f_1, ..., f_n$ are functions that have at least $n-1$ derivatives.

Definition: the Wronskian of $f_1, ..., f_n$ is

$$W(f_1, ..., f_n) = \det\left(\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2'' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}\right)$$

This is a function of x.

Theorem: Assume $y_1, ..., y_n$ are n solutions to

$$a_n(x)\frac{\mathrm{d}y^n}{\mathrm{d}x^n} + a_{n-1}\frac{\mathrm{d}y^{n-1}}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0$$

on an interval I. Then $y_1, ..., y_n$ are linearly independent if and only if $W(y_1, ..., y_n) \neq 0 \forall x \in I$.

(The above equation is an n^{th}-order homogenous linear equation.)

Moreover,
- If $W(y_1, ..., y_n) = 0$ for some $x_0 \in I$, $W(y_1, ..., y_n) = 0 \forall x \in I$.
- If $W(y_1, ..., y_n) \neq 0$ for some $x_0 \in I$, $W(y_1, ..., y_n) \neq 0 \forall x \in I$.

(In other terms, if the Wronskian is either zero or nonzero for some number in the interval, it's that everywhere else in the interval.)

If $W(y_1, ..., y_2) \neq 0 \Rightarrow \{y_1, ..., y_n\}$ is called a fundamental set of solution to the n^{th}-order homogenous linear equation. So any solution to the ODE is given as

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

(this is a general solution).

Example: Given $x^2 y'' + 7xy' + 13y = 0, y_1 = \frac{\cos(2\ln(x))}{x^3}, y_1 = \frac{\sin(2\ln(x))}{x^3}, I = (0, \infty)$, is $\{y_1, y_2\}$ a fundamental set of solutions?

Solution: We compute the Wronskian…

$$W(y_1, y_2) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} \frac{\cos(2\ln(x))}{x^3} & \frac{\sin(2\ln(x))}{x^3} \\ \frac{-2x^2\sin(2\ln(x)) - 3x^2\cos(2\ln(x))}{x^6} & \frac{2x^2\cos(2\ln(x)) - 3x^2\sin(2\ln(x))}{x^6} \end{bmatrix}\right)$$

(end of class)

Lecture 35: non-homogenous equations and reduction of order

Non-homogenous equations

Once again, recall that

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_0(x)y = g(x)$$

is an n^{th}-order ODE that is linear. The general solution is

$$y = \begin{pmatrix} \text{general solution to the} \\ \text{associated homogenous equation} \end{pmatrix} + \begin{pmatrix} \text{particular solution} \\ \text{to g(x) above} \end{pmatrix}$$

$$= c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p, c_i \in \mathbb{R}$$

Example: Consider $y''' - 6y'' + 11y' - 6y = 3x$. Do the following numbered items:

1. First, verify that $y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}, x \in (-\infty, \infty)$ is a fundamental set of solutions.

Solution: We are given a third-order linear ODE and three solutions (y_1, y_2, y_3). That means we only need to verify that these solutions are linearly independent.

The Wronskian is

$$W(y_1, y_2, y_3) = \det\left(\begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix}\right)$$

We can pick one number as a sample point in the interval (so some number $n \in (-\infty, \infty)$), and if the Wronskian is nonzero for that number, it's nonzero everywhere. Here, it's convenient to pick 0.

$$W|_{x=0} = \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\right) = 2 \neq 0 \checkmark$$

Hence $y_1(x) = e^x, y_2(x) = e^{2x}, y_3(x) = e^{3x}$ is a fundamental set of solutions to $y''' - 6y'' + 11y' - 6y = 0$.

2. Verify that $y_p = -\frac{11}{12} - \frac{1}{2}x, x \in (-\infty, \infty)$ is a particular solution.

Calculating the derivatrives, $y_p' = -\frac{1}{2}, y_p'' = 0, y_p''' = 0$. Then substituting that into the given equation,

$$0 - 6(0) + 11\left(-\frac{1}{2}\right) - 6\left(-\frac{11}{12} - \frac{1}{2}x\right) = 3x$$

Yep, this is a solution to $y''' - 6y'' + 11y' - 6y = 3x$ on the interval $(-\infty, \infty)$! We get the same right hand side when we plug it into the equation.

3. Write the general solution.

Just do the general solution to the associated homogenous equation + the particular solution:

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + \left(-\frac{11}{12} - \frac{1}{2}x\right) \text{ on } (-\infty, \infty)$$

Consider $y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$ on $(-\infty, \infty)$ and, by splitting it up into groups, the following equations:

$$\begin{cases} (1)\ y'' - 3y' + 4y = -16x^2 + 24x - 8 \\ (2)\ y'' - 3y' + 4y = 2e^{2x} \\ (3)\ y'' - 3y' + 4y = 2xe^x - e^x \end{cases}$$

The superposition principle establishes that if y_{p_1} is a solution to (1), y_{p_2} is a solution to (2), and y_{p_3} is a solution to (3), then $y_{p_1} + y_{p_2} + y_{p_3}$ is also a particular solution to the initial (big) equation.

equation	*particular solution*
(1)	$y_{p_1}(x) = -4x^2$
(2)	$y_{p_2}(x) = e^{2x}$
(3)	$y_{p_3}(x) = xe^x$

A particular solution to the given equation is then

$$y_p = -4x^2 + e^{2x} + xe^x \text{ on } (-\infty, \infty)$$

(We'll learn how to actually find the particular solutions next lecture.)

Reduction of order (§ 4.2)

Consider $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and assume we know one solution $y_1(x)$. How do we find a second solution $y_2(x)$ such that $y_2(x)$ is linearly independent from $y_1(x)$? By the method of *reduction of order.*

(We're not going over why this works — he recommends just memorizing the formula, since it's a lengthy derivation.)

Reduction of order steps:
Given $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$,

1. Convert the given equation to the standard form

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$
$$\downarrow a_2(x) \neq 0 \downarrow$$
$$y'' + P(x)y' + Q(x)y = 0$$

2. Compute

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,\mathrm{d}x}}{y_1^2(x)}\,\mathrm{d}x$$

Example: Given $x^2y'' - 3xy' + 4y = 0$, and assume $y_1(x) = x^2$ is a solution on $(-\infty, \infty)$. Find a general solution.

Solution: We apply reduction of order:

First, converting to standard form:

$$y'' - \frac{3x}{x^2}y' + \frac{4}{x^2} = 0$$

Notice that by dividing, $x^2 \neq 0 \Leftrightarrow x \in (-\infty, 0) \vee x \in (0, \infty)$

Then the second solution is

$$y_2(x) = x^2 \int \frac{e^{-\int -\frac{3}{x} \, \mathrm{d}x}}{\left(x^2\right)^2} \, \mathrm{d}x$$

$$= x^2 \int \frac{e^{\int \frac{3}{x} \, \mathrm{d}x}}{x^4} \, \mathrm{d}x$$

$$= x^2 \int \frac{e^{3 \ln|x|}}{x^4} \, \mathrm{d}x$$

We can remove the absolute value by picking an interval. Here, let's pick $x \in (0, \infty)$.

$$\overset{x \in (0,\infty)}{=} x^2 \int \frac{e^{3 \ln x}}{x^4} \, \mathrm{d}x$$

$$= x^2 \int \frac{e^{\ln x^3}}{x^4} \, \mathrm{d}x$$

$$= x^2 \int \frac{x^3}{x^4} \, \mathrm{d}x$$

$$= x^2 \int \frac{1}{x} \, \mathrm{d}x$$

$$= x^2 \ln|x| + C$$

Taking $C = 0$ and again given that we've chosen $x \in (0, \infty)$, our solution is

$$y_2 = x^2 \ln x$$

Then our general solution is $y_1 + y_2$

(end of class)

Lecture 36: homogenous linear ODEs with constant coefficients and undetermined coefficients

Homogenous linear ODEs with constant coefficients

We consider

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where a_i is a real number (constant).

(For the following, we assume its solutions are of the form $y = e^{mx}$.)

Definition: the auxiliary equation for the above equation is

$$a_n + m^n + a_{n-1} + m^{n-1} + \cdots + a_1 m + a_0 = 0$$

This is a polynomial in the variable m with degree n. Therefore the solutions to this polynomial yield n linearly independent solutions to the above differential equation — i.e., they form a fundamental set of solutions.

If $n = 2$, then the differential equation becomes

$$ay'' + by' + cy = 0$$

and then the auxillary equation is

$$am^2 + bm + c = 0$$

By the quadratic formula, the roots of this auxillary equation are $m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and $\Delta = b^2 - 4ac$ is the discriminant.

1. If $\Delta = b^2 - 4ac > 0$, then $m_1 \neq m_2$ and $m_1, m_2 \in \mathbb{R}$.
   - In this case, $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}$ is a fundamental set of solutions.
2. If $\Delta = b^2 - 4ac = 0$, $m_1 = m_2 = -\frac{b}{2a} \in \mathbb{R}$. This gives us one solution repeated twice. So becuase $y_1 = y_2$ and $m_1 = m_2$, we have $y_1 = e^{m_1 x}$. To find a second solution, we use reduction of order.
   - This gives us $y_1 = e^{m_1 x}, y_2 = x e^{m_1 x}$ as the fundamental set of solutions
3. If $\Delta = b^2 - 4ac < 0$, that means m_1, m_2 are conjugate complex numbers of the form $m_1 = \alpha + \beta i, m_2 = \alpha - \beta i$.
   - A fundamental set of solutions is then $y_1 = e^{(\alpha + \beta i)x}, y_2 = e^{(\alpha - \beta i)x}$
   - Equivalently, $y_1 = e^{\alpha x} \cos(\beta x), y_2 = e^{\alpha x} \sin(\beta x)$, which comes from Euler's formula $e^{i\theta} = \sin(\theta) + \cos(\theta)$

Example: Solve $y'' - y' - 6y = 0$.

(This is an example for (1))

Solution: The auxilllary equation is

$$m^2 - m - 6 = 0$$
$$(m+2)(m-3) = 0$$

so its roots are

$$m_1 = -2, m_2 = 3$$

Hence, the fundamental set of solutions is

$$y_1 = e^{-2x}, y_2 = e^{3x}$$

and the general solution is

$$y = C_1 e^{-2x} + C_2 e^{3x}$$

(Note: the textbook sometimes calls the general solution the "complementary solution" and use $y_c =$ ....)

Example: Solve $y'' + 8y' + 16y = 0$

(This is an example for (2))

Solution: The auxiliary equation is $m^2 + 8m + 16 = 0$, which gives us $(m+4)^2 = 0 \Rightarrow m = -4$.

Thus a fundamental set of solutions is

$$y_1 = e^{-4x}, y_2 = xe^{-4x}$$

and the general/complementary solution is

$$y_c = C_1 e^{-4x} + C_2 x e^{-4x}$$

Example: Solve $y'' + 4y' + 6y = 0$

(This is an example for case (3))

Solution: Our auxiliary equation is $m^2 + 4m + 6 = 0 \Rightarrow \Delta = 4^2 - 4(1)(6) = -8$ so then

$$m_1 = \frac{-4 + \sqrt{\Delta}}{2(1)} = \frac{-4 + \sqrt{-8}}{2} = \frac{-4 + \sqrt{(-1)(4)(2)}}{2} = \frac{-4 + 2i\sqrt{2}}{2} = \underbrace{-2}_{\alpha} + \underbrace{2\sqrt{i}}_{\beta}$$

This makes the following fundamental set of solutions:

$$y_1 = e^{-2x} \cos\left(\sqrt{2}x\right), y_2 = e^{-2x} \sin\left(\sqrt{2}x\right)$$

and the general/complementary solution

$$y_C = C_1 e^{-2x} \cos\left(\sqrt{2}x\right) + C_2 e^{-2x} \sin\left(\sqrt{2}x\right)$$

Example: Solve $y^{(5)} - 3y^{(4)} + 4y^{(3)} - 4y'' + 3y' - y = 0$

(This is a combination of multiple cases)

Solution: The auxillary equation is

$$m^5 - 3m^4 + 4m^3 - 4m^2 + 3m - 1 = 0$$

(Note: to solve cubics, review synthetic division and the $\frac{p}{q}$ thing for guessing the first root.)

Factoring that, we have

$$(m^2 + 1)(m - 1)^3 = 0$$

which gives us

$$m = \pm i, 1$$

(the root 1 has multiplicity 3, i.e. it's repeated three times)

That means that we have the following as a fundamental set of solutions:

$$y_1 = e^{0x} \cos(1x), y_2 = e^{0x} \sin(1x), y_3 = e^{1x}, y_4 = xe^{1x}, y_5 = x^2 e^{1x}$$

which in turn means the general/complementary solution is

$$y_c = C_1 \cos(x) + C_2 \sin(x) + C_3 e^x + C_4 x e^x + C_5 x^2 e^x$$

Undetermining coefficients (§ 4.4)

Consider again

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

But now we're going to focus on finding a particular solution y_p for this equation.

The idea behind undetermined coefficients is to make an educated guess about y_p. This method only works if $g(x)$ is:

- $g(x)$ constant
- $g(x)$ polynomial
- $g(x)$ is $\sin(x), \cos(x)$ (but not any of the other trig functions!)
- $g(x)$ exponential function
- $g(x)$ finite sums or products of the above

Quick example:

- $g(x) = x^2 \sin(x) + e^{2x}$ ✓ yep
- $g(x) = \tan(x)$ nope
- $g(x) = \frac{1}{x}$ nope

We will also consider if the guess y_p contains solutions that appear in y_c.

Lecture 31: undetermined coefficients and variation of parameters

Undetermined coefficients

Example: $\underbrace{y'' - 2y' - 3y}_{\substack{\text{linear with} \\ \text{constant coefficients}}} = \underbrace{4x + 5 + 6xe^{2x}}_{\substack{g(x) \text{ must be of} \\ \text{a specific type}}}$

Solution:

1. Find the complementary solution to $y'' - 2y' - 3y = 0$, which is then

$$m^2 - 2m - 3 = 0 \Rightarrow (m-3)(m+1) = 0 \Rightarrow m = 3, m = -1$$

so

$$y_c = C_1 e^{3x} + C_2 e^{-x}$$
$$C_1, C_2 \in \mathbb{R}, x \in (-\infty, \infty)$$

2. Find a particular solution y_p. We use undetermined coefficients, but first, we split the right side by "types of equations" and consider the equations

$$\begin{cases} \text{a) } y'' - 2y' - 3y = 4x + 5 \text{ (polynomial of degree 1)} \\ \text{b) } y'' - 2y' - 3y' = 6xe^{2x} \text{ (polynomial of degree 1)} * \text{(exponential } e^{2x}\text{)} \end{cases}$$

Note: Table 4.4.1 in the textbook is probably helpful, but you can't use it on the exam, so there's that

Our proposed solution for (a) is

$$y_{p_1} = Ax + B$$

Computing the derivatives of this,

$$y'_{p_1} = A, y''_{p_1} = 0$$

Substituting,

$$0 - 2(A) - 3(Ax + B) = 4x + 5$$

And then comparing coefficients,

$$\begin{cases} -3A = 4 \\ -2A - 3B = 5 \end{cases} \Rightarrow \begin{cases} A = -\frac{4}{3} \\ B = \frac{23}{9} \end{cases}$$

so we have

$$y_{p_1} - \frac{4}{3}x + \frac{23}{9}$$
$$x \in (-\infty, \infty)$$

Next, our proposed solution for (b) is

$$y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$$

And again computing derivatives

$$y'_{p_2} = Ce^{2x} + 2Ce^{2x} + 2De^{2x} = 2Cxe^{2x} + (C + 2D)e^{2x}$$

$$y''_{p_2} = 2Ce^{2x} + 4Cxe^{2x} + 2(C + 2D)e^{2x} = 4Cxe^{2x} + 4(X + D)e^{2x}$$

Substituting in (b):

$$\left(4Cxe^{2x} + 4(X + D)e^{2x}\right) - 2\left(2Cxe^{2x} + (C + 2D)e^{2x}\right) - 3\left(Cxe^{2x} + De^{2x}\right) = 6xe^{2x}$$

$$-3Cxe^{2x} + (2C - 3D)e^{2x} = 6xe^{2x}$$

Comparing coefficients,

$$\begin{cases} -3C = 6 \\ 2C - 3D = 0 \end{cases} \Rightarrow \begin{cases} C = -2 \\ D = -\frac{4}{3} \end{cases}$$

So the particular solution for (b) is

$$y_{p_2} = -2xe^{2x} - \frac{4}{3}e^{2x}$$
$$x \in (-\infty, \infty)$$

Time to put it all together! All three solutions (the complementary one and two particular ones) have a domain of $x \in (-\infty, \infty)$. We just have to add them together — hence, the solution is

$$y = y_c + y_{p_1} + y_{p_2}$$
$$x \in (-\infty, \infty)$$

Example:

$$y'' - 6y' + 9y = \underbrace{6x^2 + 2}_{\substack{\text{quadratic} \\ \text{polynomial}}} + \underbrace{-12e^{3x}}_{\substack{\text{constant } x\, * \\ \text{exponential } e}}$$

Solution: First, we find y_c:

$$y'' - 6y' + 9y = 0$$

$$m^2 - 6m + 9y = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3$$

The complementary solution is

$$y_c = C_1 e^{3x} + C_2 x e^{3x}$$
$$x \in (-\infty, \infty)$$

Now for the particular solutions: first we analyze

$$y'' - 6y' + 9y = 6x^2 + 2$$

and we propose

$$y_{p_1} = Ax^2 + Bx + C$$

and after doing computations

$$\begin{cases} A = \frac{2}{3} \\ B = \frac{8}{9} \\ C = \frac{2}{3} \end{cases}$$

so

$$y_{p_1} = \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3}$$
$$x \in (-\infty, \infty)$$

and then for the second particular solution

$$y'' - 6y' + 9y = -12e^{3x}$$

we propose

$$y_{p_1} = Ke^{3x} \xRightarrow{\text{multiply by } x} y_{p_2} = Kxe^{3x} \xRightarrow{\text{multiply by } x} y_{p_2} = Kx^2 e^{3x}$$

Solving to find K, we obtain $K = -6$, so we have

$$y_{p_2} = -6x^2 e^{3x}$$
$$x \in (-\infty, \infty)$$

Substituting this in, we end up with

$$6xe^{2x}$$

Variation of parameters

Notes on this method:

1. Method due to Lagrange
2. $g(x)$ has no restrictions (it can be any function)
3. The method applies to any linear ODE
   - But we will only cover the technique for linear ODEs with constant coefficients

The method: Consider $a_2 y'' + a_1 y' + a_0 y = g(x), a_i \in \mathbb{R}$.

1. Compute the complementary solution y_c for the homogenous equation

$$a_2 y'' + a_1 y' + a_0 y = 0$$

You can solve this with the characteristic polynomial, and you end up with

$$y_c = C_1 y_1 + C_2 y_2$$

Take y_1 and y_2 for the next step.

2. Compute the Wronskian of y_1, y_2: $W(y_1, y_2) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)$
3. Assume the particular solution is of the form $y_p = u_1 y_1 + u_2 y_2$ for some functions u_1, u_2 (which are unknown).

(end of class)

Last lecture: variation of paramerters and vibrations

Variation of parameters

Problem: we're trying to solve a second-order differential equation with constant coefficients like

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

1. Compute the complementary solution y_c (just consider the homogenous part)
   - of the form $y_c = c_1 y_1 + c_2 y_2$
2. Assume that a particular solution is of the form $y_p = u_1 y_1 + u_2 y_2$. Compute the functions u_1, u_2.
   - To do so:
     1. Compute $W(y_1, y_2) = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)$
        - *Note:* don't pick sample points
        - Compute $W_1 = \det\left(\begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}\right), W_2 = \det\left(\begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix}\right)$
        - We transform the differential equation into standard form like this:

$$f(x) = \frac{g(x)}{a_2} = y'' + \frac{a_1}{a_2} y' + \frac{a_0}{a_2} y$$

     2. Set $u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W}, u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}$
     3. Integrate with respect to x to compute u_1, u_2

Example: $y'' - 4y' + 4y = (x+1)e^{2x}$

Solution:

1. First, compute y_c:

$$m^2 - 4m + 4 = 0 \Rightarrow m = 2 \text{ (multiplicity 2)}$$

$$y_c = C_1 \underbrace{e^{2x}}_{y_1} + C_2 \underbrace{xe^{2x}}_{y_2}$$

2. The particular solution has the form $y_p = u_1 y_1 + u_2 y_2$.

To find u_1, u_2:

$$W(y_1, y_2) = \det\left(\begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{bmatrix}\right) = e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x}xe^{2x} = e^{4x}$$

The given ODE is already in standard form, so $f(x) = (x+1)e^{2x}$

$$u_1' = -\frac{y_2 f(x)}{W} = -\frac{xe^{2x}(x+1)e^{2x}}{e^{4x}} = -x(x+1) = -x^2 - x$$

Integrating u_1':

$$u_1 = \int (-x^2 - x)\,\mathrm{d}x = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

Note: we ignore the usual $+C$ coefficient of integration here

Then for u_2':

$$u_2' = \frac{y_1 f(x)}{W} = \frac{e^{2x}(x+1)e^{2x}}{e^{4x}} = x + 1$$

and then integrating to get u_2:

$$\int x + 1\,\mathrm{d}x = \frac{1}{2}x^2 + x$$

Then a particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Hence, the general solution is

$$y = y_c + y_p$$

$$y = C_1 e^{2x} + C_2 xe^{2x} + \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Example: $4y'' + 36y = \csc(3x)$

We can't solve this using undetermined coefficients because $\csc(3x) = \frac{1}{\sin(3x)}$, which is a quotient of two functions, not a product or a sum. Hence, we approach the problem using variation of parameters.

1. Compute y_c:

$$4m^2 + 36 = 0 \Leftrightarrow m^2 + 9 = 0 \Leftrightarrow m = \pm 3i$$

so

$$y_c = C_1 e^{0x}\cos(3x) + C_2 e^{0x}\sin(3x)$$

$$y_c = C_1 \underbrace{\cos(3x)}_{y_1} + C_2 \underbrace{\sin(3x)}_{y_2}$$

2. Compute y_p:

$$W(y_1, y_2) = \det\left(\begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}\right) = 3\cos^2(3x) + 3\sin^2(3x) = 3(\cos^2(3x) + \sin^2(3x)) = 3$$

Putting the differential equation in standard form:

$$y'' + 9y = \underbrace{\frac{\csc(3x)}{4}}_{f(x)}$$

Now,

$$u_1' = -\frac{y_2 f(x)}{W} = -\frac{\sin(3x)\frac{\csc(3x)}{4}}{3} = -\frac{\sin(3x)\frac{1}{\sin(3x)}}{12} = -\frac{1}{12}$$

$$u_1 = \int -\frac{1}{12}\,\mathrm{d}x = -\frac{1}{12}x$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{\cos(3x)\frac{\csc(3x)}{4}}{3} = \frac{1}{12}\frac{\cos(3x)}{\sin(3x)}$$

$$u_2 = \int \frac{1}{12}\frac{\cos(3x)}{\sin(3x)}\,\mathrm{d}x$$

Using u-substitution to solve:

$$u = \sin(3x), \mathrm{d}u = \cos(3x)3\,\mathrm{d}x, \frac{\mathrm{d}u}{3} = \cos(3x)\,\mathrm{d}x$$

$$u_2 = \frac{1}{12}\frac{1}{3}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{36}\ln|u| = \frac{1}{36}\ln|\sin(3x)|$$

Add everything together to get a general solution.

Vibrations (§ 5.1)

Picture a spring with a mass attached to the bottom. The string's natural length is l, the length added by hanging the mass is s, and the position of the mass relative to $s + l$ is $x(t)$. (Basically $x(t)$ is how far we pull the string down, and then how much it oscillates.)

Modeling the vibration of the spring yields the second order linear ODE

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx$$

where m is the mass and k is the spring constant, which can be determined by Hooke's law $F = ks$

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0$$

We can write

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \left(\sqrt{\frac{k}{m}}\right)^2 x = 0$$

Notes for the final exam

Topics to focus on

Differential equations things

1. Autonomous equations
   - Similar to 3rd exam question
2. 1st order linear ODEs
   - Make sure to put the equation in standard form, then do the integrating factor
3. Exact equations
   - Know about the criterion for exactness (partials of M and N I think?)
4. Making non-exact equations into exact equations
   - Again, the key thing to do is using an integrating factor
   - Also make sure to again check for exactness
5. Homogeneous equations with constant coefficients
   - Get rid of the coefficient of y'' by dividing everything by it
   - If the discriminant $\sqrt{b^2 - 4ac}$ is negative, you know you can't factor it and the solutions are negative
6. Undetermined coefficients
7. Reduction of order
   - Pair types of expressions and solve
   - Only applies to 2nd-order linear ODEs in standard form
   - You need a given solution y_1, and then you can find a second solution y_2
8. Variation of parameters
   - Solve the homogeneous part first, then find a particular solution
   - A question might give you the homogeneous solutions and ask for a particular one
9. Solve a differential equation (using any of the above strategies) + initial value problems + find the interval of solution

Linear algebra things

10. Eigenvalues & eigenvectors
    - e.g. what is the corresponding eigenvector
11. Spaces associated with a matrix $\mathrm{col}(A), \mathrm{row}(A), \mathrm{null}(A)$
12. Orthogonal complements for subspaces and the above spaces
    - e.g. what is the orthogonal complement of the column/row/null space of A?
    - what is the dimension of the column space given a matrix's dimensions and its rank?
13. Least squares solution
    - Normal equation
    - Unique solution iff A has linearly independent columns
14. Change of basis matrix
    - $P_{C \leftarrow B}$
    - Practice this with polynomials
15. Orthogonal projection
    - Vector onto a subspace W: $\mathrm{proj}_W(v) = \mathrm{proj}_{v_1}(v) + \cdots + \mathrm{proj}_{v_k}(v)$
    - $W = \mathrm{span}(v_1, ..., v_k)$ and $\{v_1, ..., v_k\}$ are orthogonal basis vectors of W
    - But to use this, you have to make sure that the basis vectors are orthogonal! If not, use Gram-Schmidt to orthogonalize them
16. Adjoint matrix and formula for inverse
    - $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$
    - Allows you to compute one entry of the inverse matrix without computing the whole matrix
17. Linear transformations (rank and nullity)

18. Similarity and diagonalization
    - Similarity: $A \sim B \Leftrightarrow PAP^{-1} = B$
    - Diagonalization: $A \sim D \Leftrightarrow PAP^{-1} = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Note: no vibrations or mixing problems as of the time of writing

Sample questions

DiffEQ sample questions

1) Consider the IVP $\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = y^2(4-y^2) \\ y(1) = \frac{\pi}{2} \end{cases}$. Compute $\lim_{x \to \infty} y(x)$.

I think the stability thing

2) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 3y, y(0) = \frac{1}{3}$.

This is a first-order linear ODE. First, put it in standard form, then find an integrating factor to solve. (Then apply the initial value to find the constant.)

3) Is $(2xy^2 - 3)\,\mathrm{d}x + (2x^2y + 4)\,\mathrm{d}y = 0$ exact?
- If so, solve
- If not:
  - Can it be made exact?
  - What would an integrating factor be?

It's in differential form, so apply the criterion of exactness. (Probably see class notes for this)

If not already exact, try to make P or Q a function of just x or just y by dividing by the other

Then, you can get $\mu = e^{\int P(x)\,\mathrm{d}x}$ or $\mu = e^{\int P(y)\,\mathrm{d}y}$

4) Consider $x^2y'' + 2xy' - 6y = 0$. A solution is $y_1 = x^2$. Find a second solution y_2 so that $\{y_1, y_2\}$ is a fundamental set of solutions.

Apply reduction of order. This is a second-order homogeneous linear ODE.

5) Consider $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{\frac{3}{2}}$. A fundamental set of solutions to the homogenous part is

$$y_1 = x^{\frac{1}{2}}\cos(x), y_2 = x^{\frac{1}{2}}\sin(x)$$

Find a particular solution y_p to the given ODE.

Use variation of parameters (write it in standard form).

Linear sample questions

6) Let A be a 121×257 matrix. If $\operatorname{rank}(A) = 95$, what are $\dim\left(\operatorname{row}(A)^\perp\right)$ and $\dim\left(\operatorname{col}(A)^\perp\right)$?

We know that always

$$\dim(\operatorname{row}(A)) = \dim(\operatorname{col}(A)) = \operatorname{rank}(A) = 95$$

We can consider the 121 rows of A as vectors. Each of these rows live in $\mathbb{R}^{257}$. So the row space $\operatorname{row}(A)$ is inside $\mathbb{R}^{257}$.

By the rank and nullity theorem, $\dim\left(\operatorname{row}(A)^\perp\right) = 257 - 95 = 162$

Now for the columns. The columns live in $\mathbb{R}^{121}$, so $\dim\left(\operatorname{col}(A)^\perp\right) = 121 - 95 = 26$

7) Compute the orthogonal projection of V onto W, where

$$V = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

The basis vectors aren't orthogonal, so apply Gram-Schmidt and then the formula for orthogonal projection

8) Let $T : P_3 \to P_3$ be the linear transformation given by

$$T(p(x)) = xp(0) + x^2 p''(x)$$

- Find the rank and nullity of T
- Find $[T]_{E \leftarrow E}$, where E is the standard basis of P_3.

(end of class)

Joe’s section

Here marks ***Joe’s*** famous circle - the first of many great accomplishments and contributions he made to this document

Here is an example of how this addition completely changed the game: